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Self-correspondences of the generic fibre of Lefschetz pencils and the Leray filtration

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Abstract

In this paper we give a detailed analysis of the interaction between homological self-correspondences of the general fibre $\mathcal{Y}/k(t)$ of the Lefschetz fibration $\rho : \tilde{X} \to \mathbb{P}^1$ of a Lefschetz pencil on a smooth projective variety X/k, and the Leray filtration of ρ. We derive the result that, if the standard conjecture $B(\mathcal{Y})$ holds, then the operator $\Lambda_X - p_X^{n+1}$ is algebraic, where p_X^{n+1} is defined as the inverse of L on $LP^{n-1}(X)$ and 0 on $L^k P^j(X)$ for $(1, n-1) \neq (k, j)$; in the course of our proof we see that, under the above assumption, the Künneth projectors π_X^i for $i \neq n-1, n, n+1$ are algebraic.

Keywords: Standard conjectures; Algebraic cycles; Motives

1. Introduction

Let k be an arbitrary field. All varieties involved are assumed to be smooth and projective, unless otherwise stated. The notations on correspondences that we adopt are those of Kleiman [16, 1.3], Jannsen [13], Scholl [19]. We fix a prime $\ell \neq \operatorname{char} k$, and an algebraic closure $k \subset \bar{k}$; we then fix an isomorphism $\mathbb{Z}_\ell(1) \approx \mathbb{Z}_\ell$, and go on with the 'heresy' [9] except when keeping track of Tate twists is useful for the reader. We denote the *étale* cohomology groups $H^i(X \times_k \bar{k}, \mathbb{Q}_\ell)$ by $H^i(X)$.

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Let X be a smooth projective variety of dimension n over k. Following Kleiman [16], we denote the trace (or orientation) map by $\langle\rangle : H^*(X) \to \mathbb{Q}_\ell$ and the Poincaré duality pairing by $\langle,\rangle : H^i(X) \otimes H^{2n-i}(X) \to \mathbb{Q}_\ell$.

Fix now a very ample line bundle $\mathcal{L}$, giving an immersion in $\mathbb{P}^N$. Let Y be a smooth hyperplane section; we write $\xi_X := [Y] \in H^2(X)(1)$. Let L_X (L when not misleading) be the Lefschetz operator $L_X x = [Y] \wedge x$, where $\wedge$ denotes the cup-product in $H^*(X)$. $A^*(X)$ will denote the graded ring of algebraic cycles modulo H-homological equivalence with coefficients over $\mathbb{Q}$, and $A^{n+*}(X \times X)$ will denote the ring of homological correspondences tensored with $\mathbb{Q}$ taking $\circ$ as the product. We now recall the definition of $\circ$: let X_i be smooth projective varieties of respective dimensions n_i. If $\alpha \in A^{n_1+r}(X_1 \times X_2)$, $\beta \in A^{n_2+s}(X_2 \times X_3)$ then

$$\beta \circ \alpha = p_{13*}\big(p_{12}^*\alpha \bullet p_{23}^*\beta\big) \in A^{n_1+r+s}(X_1 \times X_3).$$

The operation $\circ$ satisfies all the good properties [8] and makes $A^{n+*}(X \times X)$ into a $\mathbb{Q}$-algebra. By Poincaré duality, we may view α as a linear operator $H^*(X_1) \to H^*(X_2)$ via $\alpha(x) = p_{2*}^{12}(\alpha p_1^{12*}(x_1))$ for $x_1 \in H^*(X_1)$ and likewise with β, and the circle product $\beta \circ \alpha$ agrees with the composition of operators. The degree of a correspondence $u \in CH^{\dim X+r}(X \times X')$ is defined to be r as usual [8], and the cohomological degree of u, i.e. the degree of u as an operator in cohomology $H^*(X) \to H^*(X')$ is $2r$, as $uH^i(X) \subset H^{i+2r}(X')$.

Given a subspace V of $H^n(X)$ such that the cup-product restricts to a non-degenerate bilinear form on V, we denote by e_V the orthogonal projection onto V with respect to the Poincaré duality pairing.

Another notation we shall adopt is the following. Whenever we consider the motive modulo homological equivalence defined by a smooth projective variety X we will denote it by $h(X)$, and the Chow motive of X (i.e. modulo rational equivalence with rational coefficients, see [19]) will be henceforth denoted by $h_{rat}(X)$. Thus $CH^{n+*}(X \times X)_{\mathbb{Q}} = \operatorname{End} h_{rat}(X)$ and $A^{n+*}(X \times X) = \operatorname{End} h(X)$.

The Hard Lefschetz theorem [5] states that the maps

$$L^{n-i} : H^i(X) \to H^{2n-i}(X)$$

are isomorphisms (henceforth called *Lefschetz isomorphisms*) (as a result, the odd Betti numbers b_{2i-1} are always even). One can thereby define the primitive subspaces $P^i(X) = \operatorname{Ker} L^{n-i+1} \cap H^i(X)$, and one has a Lefschetz decomposition of $H^*(X)$: $H^i(X) = \bigoplus L^j P^{i-2j}(X)$. Let $x = \sum L^j x_{i-2j}$ be the Lefschetz decomposition of $x \in H^i(X)$. Denote $i_1 = \max\{i-n, 1\}$. We define the following operators of degree -2:

$$\Lambda x = \sum_{j \geqslant i_1} L^{j-1} x_{i-2j},$$

$${}^c\Lambda x = \sum_{j \geqslant i_1} j(n-i+j+1) L^{j-1} x_{i-2j}.$$

We denote the Künneth projectors $H^*(X) \twoheadrightarrow H^i(X) \hookrightarrow H^*(X)$ by $\pi_X^i = \pi^i$. We define the operator of degree 0

$$H = H_X = \sum_{i=0}^{2n} (n-i)\pi_X^i.$$

The following operators are also essential: for $x = \sum L^j x_{i-2j} \in H^i(X)$, $p^k x = \delta_{i,k} x_k$, when $i \leqslant n$, and $p^k x = \delta_{i,k} x_{2n-k}$ for $k > n$; it is clear that p^i is a projector for $i \leqslant n$. Whenever we have polarised varieties X_i, we will consider the induced polarisation on $X_1 \times X_2$, and so $L_{X_1 \times X_2} = L_{X_1} \otimes 1 + 1 \otimes L_{X_2}$. We will do likewise when we have an inclusion; for instance, let ι: $Y \subset X$ denote an inclusion of a smooth hyperplane section. Then $\xi_Y = \iota^* \xi_X$ and $L_X = \iota_* \iota^*$, $L_Y = \iota^* \iota_*$. We define the *vanishing cohomology* of Y (or of $Y \subset X$ proper) by $V(Y) = \operatorname{Ker} \iota_* | H^{n-1}(Y) \subset H^{n-1}(Y)$, with Y as above.

We recall the following result:

Proposition 1.1. *(See Kleiman [16, 1.4.6], [1].) The operators ${}^c\Lambda, L, H$ are an $\mathfrak{sl}_2$-triple; in other words, the following identities hold:*

$$\left[{}^c\Lambda, L\right] = H, \qquad [H, L] = -2L, \qquad \left[H, {}^c\Lambda\right] = 2\,{}^c\Lambda.$$

The following conjecture was stated by Grothendieck, and is one of his *standard conjectures* [9,16]:

B(X). The operator Λ is induced by an algebraic cycle; equivalently [16, Prop. 2.3], all the operators in the $\mathfrak{sl}_2$-triple $({}^c\Lambda, L, H)$ are algebraic.

The conjecture $B(X)$ is known for curves, surfaces, generalised flag varieties, abelian varieties and is stable under products and smooth hyperplane sections [16]. We will therefore assume that $n \geqslant 3$. For a discussion on this form of the conjecture—regarding the field of definition—see 7.3. Another standard conjecture of Grothendieck, weaker than $B(X)$ (Kleiman [16, 2.4], [9]), regards the algebraicity of the Künneth projectors (again, we refer to 7.3):

C(X). The Künneth projectors π^i are algebraic for all $i = 0, \dots, 2n$.

For finite fields $C(X)$ is known [15]. The following remark is included for completeness.

Remark 1.2. Let X be defined over a field k; assume that $C(X \times_k \bar{k})$ holds. In the case when k is perfect, one can easily see that the operators $\pi^i : H^*(X) \to H^*(X)$ yield Galois-invariant classes in $H^{2n}(X \times X)(n)$, and one may obtain from a given representative Z_1 of π^i on $X_{\bar{k}} \times X_{\bar{k}}$ a k-defined algebraic cycle with rational coefficients (i.e. a cycle on X) also representing π^i (averaging over $Gal(L|k)$ with $L|k$ Galois finite such that Z_1 is defined over L). The general case follows since for $k_1|k$ finite and purely inseparable, the map induced by base change $CH^*(X)_{\mathbb{Q}} \to CH^*(X_{k_1})_{\mathbb{Q}}$ is an isomorphism.

Let X be defined over a field k, and Y be as above. Assume that both X, $[Y]$ are defined over a subfield $k_0 \subset k$. If the operator $\Lambda_X \in H^*(X \times X)$ is algebraic, i.e. if $B(X \times_k \bar{k})$ holds, we conclude as above that the correspondence given by Λ_X is represented by an algebraic cycle with rational coefficients defined over k_0, which implies $B(X)$.

The main result of this paper states as follows.

Main Theorem. *Let X be smooth projective of dimension $n \geqslant 3$. Assume the conjecture $B(\mathcal{Y})$ for the generic fibre $\mathcal{Y}$ of a Lefschetz pencil of X. Then the operator $\Lambda_X - p_X^{n+1}$ is algebraic.*

The following partial result on $C(X)$ is proven in Proposition 7.1 as a stepping stone (note that the Main Theorem includes this statement, see 7.2):

Partial result on C(X). Assume $B(\mathcal{Y})$ for $\mathcal{Y}$ as above. Then the Künneth projectors π_X^i are algebraic for all $i \neq n-1, n, n+1$.

We start with the algebraic cycle class $\Lambda_{\mathcal{Y}}$ on the generic fibre $\mathcal{Y}/k(t)$ of a Lefschetz fibration of X, satisfying condition **(A)** of Section 4 (Katz [6, XVIII.5.3]), $\rho : \tilde{X} \to \mathbb{P}^1$. In our proof, we pay special attention to the correspondences supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ (see 4.1), which turn out to preserve the Leray filtration of ρ under **(A)**, as will be seen in Proposition 4.15. Whenever we have an algebraic class u in $A^{n-1+r}(\mathcal{Y} \times \mathcal{Y})$, a **lifting** (or **extension**) of u will denote a class supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ (of codimension $n-1+r$ in this divisor) with a representative Z that yields u after restriction to the generic fibre of $\mathbb{P}^1$ and taking its cohomology class; a study of the interaction of correspondences supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ and the Leray filtration of ρ is provided. A proof of the Main Theorem is provided for every Lefschetz fibration ρ, though condition **(A)** makes things a bit easier and renders the piece $H^1(R^{n-1}\rho_* \mathbb{Q}_\ell)$ easier to understand.

The consequences of establishing the full conjecture $B(X)$ for general X would be very remarkable. Not only would this yield a satisfactory category of pure motives in characteristic zero, but as shown by Y. André [1] it would imply the Variational Hodge Conjecture, hence the Hodge conjecture for arbitrary products of the form $A \times X_1 \times \cdots \times X_m$, where A is an abelian variety and X_i are K3 surfaces [1].

2. General results

The results in this section need no more background than Kleiman [16]. For the sake of completeness, we include the proof of the following lemma.

Lemma 2.1. *Let $u \in A^{n+r}(X \times X)$ be a correspondence of degree r on X. The following identity holds:*

$$[H, u] = -2r \cdot u.$$

Proof. One has $u\pi^i = \pi^{i+2r}u$, hence

$$uH = \sum (n-i)u\pi^i = \sum (n-i)\pi^{i+2r}u = Hu + 2r \cdot u.$$

Isolating yields $[H, u] = -2r \cdot u$ as desired. $\square$

Lemma 2.2. *Let $f : X' \to X$ be a generically finite, surjective morphism of smooth projective varieties. Assume that $C(X')$ holds; then $C(X)$ holds.*

Proof. The lemma follows readily from the identity $\pi_X^i = \frac{1}{\deg(f)} f_* \pi_{X'}^i f^*$. $\square$

Lemma 2.3. *If $j_1 + 2i_1 + j_2 + 2i_2 = 2n$, then the pieces $L^{i_1} P^{j_1}(X)$ and $L^{i_2} P^{j_2}(X)$ are orthogonal for $j_1 \neq j_2$. Let $0 \leqslant i \leqslant n$; then the operator p^i is a projector, and p^{2n-i} is a symmetric operator characterised by $p^{2n-i} : L^{n-i} P^i(X) \to P^i(X)$ is given by the inverse of the Lefschetz isomorphism L^{n-i}, and $p^{2n-i} L^j P^k(X) = 0$ if $(j,k) \neq (n-i, i)$. The following identities hold:*

$$p^{2n-i} L^{n-i} = p^i, \qquad L^{n-i} p^{2n-i} = {}^t p^i.$$

Proof. The first assertion implies the rest of the lemma. Suppose $j_2 > j_1$; then

$$L^{i_2} P^{j_2}(X) \wedge L^{i_1} P^{j_1}(X) = L^{i_1+i_2} P^{j_1}(X) \wedge P^{j_2}(X) = 0;$$

indeed, consider a smooth linear section $\kappa : W \hookrightarrow X$ of codimension $(i_1 + i_2)$. Then $\kappa_* \kappa^*(P^{j_1}(X) \wedge P^{j_2}(X)) = 0$, since $j_2 > \frac{j_1+j_2}{2} = \dim W$, hence $\kappa^* P^{j_2}(X) = 0$ by [16, 1.4.7], [6, Exp. XVIII (5.2.4)]. This proves the assertion. □

Lemma 2.4. *The operators L, Λ and ${}^c\Lambda$ are symmetric.*

Proof. L, Λ are symmetric and ${}^tH = -H$. We thus have two $\mathfrak{sl}_2$-triples, $({}^c\Lambda, L, H)$ and $({}^t({}^c\Lambda), L, H)$. Using Bourbaki [2, Ch. 11] yields ${}^c\Lambda = {}^t({}^c\Lambda)$. □

Proposition 2.5.

1. *The following non-commutative rings of operators are equal:*

$$\mathbb{Q}\langle L, \Lambda\rangle = \mathbb{Q}\big\langle L, {}^c\Lambda\big\rangle = \mathbb{Q}\big\langle L, p^n, \ldots, p^{2n}\big\rangle.$$

2. *$B(X)$ holds if and only if, for all $i < n$, the inverse*

$$\theta^i : H^{2n-i}(X) \to H^i(X)$$

to the Lefschetz isomorphism $L^{n-i} : H^i(X) \xrightarrow{\sim} H^{2n-i}(X)$ is induced by an algebraic correspondence for $i < n$.

Proof. The first assertion follows from Kleiman [16, Prop. 1.4.4]. The second is proved in [16, Prop. 1.4.4], 2.3 (see also [17, Th. 4.1]). □

The morphisms ι^*, ι_* are well-behaved with respect to the Lefschetz decompositions of X, Y (see Kleiman [16, Prop. 1.4.7]). The following two lemmas relate the operators Λ_X, Λ_Y.

Lemma 2.6. *The following identity holds:*

$$\iota^* \Lambda_X = \Lambda_Y \iota^* + \sum_{j=n+1}^{2n-2} \iota^* L^{j-n-1} p_X^j. \tag{1}$$

Proof. Here we use [16, Prop. 1.4.7] constantly. The operators $\iota^* \Lambda_X$ and $\Lambda_Y \iota^*$ agree on $H^i(X)$ for $i \leqslant n$. It is easy to see that $\iota^* \Lambda_X - \Lambda_Y \iota^* | L^{i-n+1} H^{2n-i-2}(X) = 0$. The equality $\iota^* \Lambda_X -$

$\Lambda_Y \iota^* = \sum_{i=n+1}^{2n-2} \iota^* L^{j-n-1} p_X^j$ thus holds for every piece $L^r P^s(X)$ of $H^*(X)$, which proves the lemma. □

The following is but a rephrasing of Kleiman [16, Prop. 2.12].

Lemma 2.7. *Notations as above. $B(X)$ holds iff $\iota^* \Lambda_X$ is algebraic.*

Proof. We reproduce the proof in [16]. First note that ${}^t(\iota^* \Lambda_X) = \Lambda_X \iota_*$. Assume that $\iota^* \Lambda_X$ is algebraic; then $\Lambda_Y = \iota^* \Lambda_X^t(\iota^* \Lambda_X) = \iota^* \Lambda_X^2 \iota_*$ is algebraic and, for every $i < n$, the map

$$\theta^i : H^{2n-i}(X) \xrightarrow{\iota^* \Lambda_X} H^{2n-2-i}(Y) \xrightarrow{\Lambda_Y^{n-1-i}} H^i(Y) \xrightarrow{\Lambda_X \iota_*} H^i(X) \tag{2}$$

is an algebraic inverse to L^{n-i}. This proves $B(X)$ by Proposition 2.5. □

The following will be used in Section 5.

Lemma 2.8. *The conjecture $C(X)$ holds if and only if the semisimple operator H is algebraic.*

Proof. The result follows readily from the identities

$$[\Delta_X] = id_{H^*(X)} = \sum \pi^i \quad \text{and} \quad H^r = \sum (n-i)^r \pi^i \quad \text{for } r \in \mathbb{N}. \qquad \square$$

3. The cohomology of Lefschetz pencils

For the basic results and the tone of this section we follow Katz [6, Exp. XVIII]; we assume k to be algebraically closed. For X an n-dimensional smooth projective variety and $X \hookrightarrow \mathbb{P}^N$ a suitable projective embedding (given by the line bundle $\mathcal{L}_X$), there exists a line $L \subset (\mathbb{P}^N)^\vee$ cutting the dual variety $X^\vee$ of $X \subset \mathbb{P}^N$ transversally; L is then called a **Lefschetz pencil**. A basic property of L is that, for every hyperplane $t \in L$, $X_t = X \cap H_t$ is either smooth or has a unique singular point which is an ordinary double point. The base locus of L in X will be denoted by Δ, and for any two $t_1 \neq t_2 \in L$ one has a transversal intersection $X_{t_1} \cap X_{t_2} = \Delta$. Thus Δ is smooth of dimension $n-2$: for any smooth member $Y = X_t$ as above, we will denote the canonical inclusion by $h : \Delta \hookrightarrow Y$. If $\tilde{X}$ denotes the blowing-up of X centred at Δ, projection induces a map $X - \Delta \to \mathbb{P}^1 \cong L$ which induces a fibration (henceforth called a **Lefschetz fibration**, or **Lefschetz pencil** by *abus de langage*):

$$\rho : \tilde{X} \to \mathbb{P}^1.$$

We denote by f the blowing-up map $f : \tilde{X} \to X$. The full blow-up diagram will be denoted as follows:

$$\begin{tikzcd} \tilde{\Delta} \arrow[r, "i"] \arrow[d, "g"] & \tilde{X} \arrow[d, "f"] \\ \Delta \arrow[r, "j"] & X, \end{tikzcd} \tag{3}$$

where j is the canonical inclusion $\Delta \subset X$. $\tilde{\Delta}$ is the exceptional divisor and, since $N_{\Delta/X} \cong \mathcal{O}_X(-1)^{\oplus 2}$, g is a trivial projective bundle; $\tilde{j}$ denotes the inclusion $\tilde{\Delta} \subset \tilde{X}$. We describe the cohomology of $\tilde{X}$ in the following proposition.

Proposition 3.1. *(See Katz [6, Exp. XVIII Prop. 4.2].) Notations and assumptions being as above. Then:*

(i) *the following homomorphisms are mutual inverses:*

$$H^\bullet(\tilde{X}) \xrightarrow{f_* \oplus g_* i^*} H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1)$$

and

$$H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1) \xrightarrow{f^* + i_* g^*} H^\bullet(\tilde{X}).$$

(ii) *Transport of structure via the above isomorphisms endows $H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1)$ with a structure of algebra, which expresses cup-product on $\tilde{X}$ as follows. For $a, b \in H^\bullet(X)$, $x, y \in H^{\bullet-2}(\Delta)(-1)$ one has:*
$(0 \oplus x) \wedge (0 \oplus y) = -j_*(xy) \oplus 2L_\Delta xy,$
$(a \oplus 0) \wedge (b \oplus 0) = ab \oplus 0,$
$(a \oplus 0) \wedge (0 \oplus y) = 0 \oplus j^*(a)y,$
$(0 \oplus x) \wedge (b \oplus 0) = 0 \oplus xj^*(b).$

The Poincaré duality pairing is expressed as follows in terms of the above decomposition. If $x \oplus y \in H^i(\tilde{X})$, $x' \oplus y' \in H^{2n-i}(\tilde{X})$, then

$$\langle x \oplus y, x' \oplus y' \rangle_{\tilde{X}} = \langle x, x' \rangle_X - \langle y, y' \rangle_\Delta.$$

Let $\iota : Y \hookrightarrow X$ denote the canonical inclusion of a smooth hyperplane section Y in X. If $Y = X_t$ is a smooth fibre $\rho^{-1}(t)$ of ρ, let $k : Y \hookrightarrow \tilde{X}$ denote the canonical inclusion. The following result expresses the cohomology of k_* and k^* in terms of Proposition 3.1.

Proposition 3.2. *(See Katz [6, Exp. XVIII 5.1.1].) Notations and assumptions as above; the restriction homomorphism is expressed by*

$$k^* = \iota^* + h_* : H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1) \to H^\bullet(Y)$$

and the Gysin homomorphism has the expression

$$k_* = \iota_* \oplus -h^* : H^{\bullet-2}(Y)(-1) \to H^\bullet(X) \oplus H^{\bullet-2}(\Delta)(-1).$$

Since we will deal with the Lefschetz theory of both X and $\tilde{X}$, the following discussion will help prove our Main Theorem.

Choice of $\mathcal{L}_{\tilde{X}}$. The line bundle $\mathcal{L}_N = f^*\mathcal{L}_X^{\otimes N} \otimes \mathcal{O}_{\tilde{X}}(-\tilde{\Delta})$ is very ample on $\tilde{X}$ for $N \geqslant 2$ (it suffices to check the case $X = \mathbb{P}^n$). For $m \geqslant 1$, we choose the polarisation $\mathcal{L}_{\tilde{X}} := \mathcal{L}_{m+1}$ on $\tilde{X}$ (remember that $\mathcal{L}_X$ was very ample).

Proposition 3.3. *Consider the polarisation on $\tilde{X}$ given by the divisor class $\xi_{\tilde{X}} = c_1(\mathcal{L}_{m+1}) = m\cdot f^*\xi_X + \rho^*([t])$ for $t \in \mathbb{P}^1$ a regular value of ρ (not necessary). Let $L_{\tilde{X}}$ be the Lefschetz operator of this polarisation. One also has $f^*\xi_X = \xi_X \oplus 0$ and $f^*(\xi_X) \wedge (x \oplus y) = L_X x \oplus L_\Delta y$. In terms of the decomposition of Proposition* 3.1, *$L^r_{\tilde{X}}$ is expressed as follows:*

$$L^r_{\tilde{X}}(x \oplus 0) = m^{r-1}(m+r)L^r x \oplus -r\cdot m^{r-1} L^{r-1}_\Delta j^* x,$$

and

$$L_{\tilde{X}}(0 \oplus y) = r\cdot m^{r-1} L^{r-1}_X j_* y \oplus m^{r-1}(m-r)L^r_\Delta y.$$

Proof. Using Propositions 3.1, 3.2 we obtain the following:

$$\xi_{\tilde{X}} = f^*\xi_X = [Y] \oplus 0, \qquad [\tilde{\Delta}] = 0 \oplus 1_\Delta, \qquad \big[\rho^*(t)\big] = [Y] \oplus -1_\Delta,$$

and $c_1(\mathcal{L}_{\tilde{X}}) = (m+1)\cdot[Y] \oplus -1_\Delta$. Using $\rho^*(t)^2 = 0$ we derive

$$\xi^r_{\tilde{X}} = m^r f^*\xi^r_X + r\cdot m^{r-1} f^*\big(\xi^{r-1}_X\big)\cdot\rho^*(t) = (m+r)m^{r-1}\xi^r_X \oplus -r\cdot m^{r-1}\xi^r_\Delta.$$

The proposition now follows from Proposition 3.1(ii). □

4. The Leray filtration of a Lefschetz pencil

Assume $k = \bar{k}$ as in the previous section. Choose a Lefschetz pencil on X, denoted by $\rho : \tilde{X} \to \mathbb{P}^1$. The Leray filtration of ρ turns out to degenerate at E_2, as in the case of smooth projective morphisms.

Condition **(A)** of Katz [6, Exp. XVIII, 5.3] will be important in our construction of the relative projectors and the establishing of some isomorphisms, but not essential in our proof of our Main Theorem. We set the framework first.

Let ν: $\mathcal{U} \subset \mathbb{P}^1$ be contained within the smooth locus of ρ. We have the adjunction morphisms

$$R^i\rho_*\mathbb{Q}_\ell \to \nu_*\nu^* R^i\rho_*\mathbb{Q}_\ell \tag{4}$$

for all $0 \leqslant i \leqslant 2n-2$.

Lemma 4.1. *(See [6, XVIII, 5.3].) The map* (4) *is an isomorphism for $i \neq n$, and is an epimorphism for $i = n$ whose kernel $\mathcal{K}$ is a skyscraper sheaf supported on the singular values of ρ (and independent of $\mathcal{U}$). As a result, the maps*

$$L^{n-1-i} : R^i\rho_*\mathbb{Q}_\ell \to R^{2n-2-i}\rho_*\mathbb{Q}_\ell$$

are isomorphisms for $i \leqslant n-3$, and the corresponding map for $i = n-2$ is an isomorphism if and only if $H^0(\mathcal{K}) = 0$.

Condition (A). (See [6, XVIII, 5.3].) Let ν: $\mathcal{U} \subset \mathbb{P}^1$ be contained within the smooth locus of ρ. The adjunction morphisms

$$R^i\rho_*\mathbb{Q}_\ell \to \nu_*\nu^*R^i\rho_*\mathbb{Q}_\ell$$

are isomorphisms for all $0 \leqslant i \leqslant 2n-2$ (independent of $\mathcal{U}$).

An immediate application of the weak Lefschetz theorem yields the first assertion of the following lemma.

Lemma 4.2. *(See [6, Exp. XVIII Lemma 5.4, Th. 6.3, Cor. 6.4]; [5].) If the Lefschetz pencil ρ satisfies condition* **(A)***, then the sheaves $R^i\rho_*\mathbb{Q}_\ell$ are constant for $i \neq n-1$. If $n = \dim X$ is even or $\operatorname{char} k \neq 2$, for a sufficiently high multiple $\mathcal{L}^N$ of a given polarisation $\mathcal{L}$ of X, any Lefschetz pencil associated to $\mathcal{L}^N$ will satisfy* **(A)***.*

Examples.

1. An easy example of a Lefschetz fibration ρ in which **(A)** holds is a smooth Lefschetz fibration. The ℓ-adic sheaf $R^n\rho_*\mathbb{Q}_\ell$ is constant in this case.
2. (See [6, pp. 311–312].) This is an example by Deligne where **(A)** does not hold. Take X to be a smooth quadric in $\mathbb{P}^3$, and consider a pencil of planes whose axis intersects X transversally. The sections of X by this pencil are plane conics, and every singular fibre X_{s_0} of ρ will be a pair of lines on a $\mathbb{P}^2$. Thus $H^2(X_t) \cong \mathbb{Q}_\ell$ for X_t a smooth member of ρ, and $H^2(X_{s_0}) \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$ for X_{s_0} a singular member (which has two irreducible components, hence the rank of H^2). Thus the sheaf $R^2\rho_*\mathbb{Q}_\ell$ has non-constant rank, so ρ does not satisfy **(A)**.

Theorem 4.3. *(See [6, 5.6, 5.6.8]; [4, Sec. 2]; [5].) For a Lefschetz pencil $\rho : \tilde{X} \to \mathbb{P}^1$, the Leray spectral sequence*

$$E_2^{i,j} = H^i\big(\mathbb{P}^1, R^j\rho_*\mathbb{Q}_\ell\big) \quad \Rightarrow \quad H^{i+j}(\tilde{X})$$

degenerates at E_2. For $k : Y = X_t \hookrightarrow \tilde{X}$ the inclusion map of a smooth fibre, the Leray filtration of ρ can be interpreted as follows:

1. *$Gr^0_{F_\rho} H^i(\tilde{X}) = H^0(\mathbb{P}^1, R^i\rho_*\mathbb{Q}_\ell)$;*
2. *$F^2_\rho H^*(\tilde{X}) = \operatorname{Im} k_*$; one has an isomorphism $F^2_\rho H^i(\tilde{X}) = H^2(\mathbb{P}^1, R^{i-2}\rho_*\mathbb{Q}_\ell) \cong k_*H^{i-2}(Y)$. $F^1_\rho H^i(\tilde{X}) = F^2H^i(\tilde{X})$ for $i \neq n$, and the restriction map induced by $\mathcal{Y} \subset \tilde{X}$ can be described as follows:*

   $$H^i(\tilde{X}) \twoheadrightarrow H^0\big(R^i\rho_*\mathbb{Q}_\ell\big) \twoheadrightarrow H^0\big(\nu_*\nu^*R^i\rho_*\mathbb{Q}_\ell\big) \cong H^i(\mathcal{Y})^{\pi_1(\mathcal{U},\bar{\eta})},$$

   where the second homomorphism is as described in (4) *for a fixed open subset $\mathcal{U} \subset \mathbb{P}^1$ within the smooth locus of ρ and $\bar{\eta}$ a geometric generic point of $\mathbb{P}^1$. For $i = n$, $\operatorname{Ker} k^* \supset F^1H^n(\tilde{X})$ with equality if and only if* **(A)** *holds.*

The piece $F^2_\rho H^*(\tilde{X})$ thus coincides with the image of the Gysin homomorphism

$$k_* : H^{*-2}(Y) \to H^*(\tilde{X}),$$

where Y is a smooth fibre of ρ (as above), so $\operatorname{Im} k_*$ is independent of the smooth fibre chosen.

Denote by L the operator in $H^*(\tilde{X})$ given by $L\bullet = f^*\xi_X \wedge \bullet$ (see Proposition 3.3), which has the following expression in terms of Proposition 3.1:

$$L(x \oplus y) = L_X x \oplus L_\Delta y. \tag{5}$$

Remark 4.4. Condition **(A)** induces a Lefschetz theory on the ℓ-adic sheaves $R^i\rho_*\mathbb{Q}_\ell$. One has Lefschetz isomorphisms

$$L^{n-1-i} : R^i\rho_*\mathbb{Q}_\ell \simeq \nu_*\nu^* R^i\rho_*\mathbb{Q}_\ell \to \nu_*\nu^* R^{2n-2-i}\rho_*\mathbb{Q}_\ell \simeq R^{2n-2-i}\rho_*\mathbb{Q}_\ell,$$

where ν: $\mathcal{U} \subset \mathbb{P}^1$ is such that ρ is smooth on $\mathcal{U}$. In general, whether **(A)** holds or not, we denote by $\mathcal{P}^i_\rho = \ker L^{n-i}$ the primitive cohomology sheaves, and use the alternative notation $\mathcal{R}^i$ for the sheaves $R^i\rho_*\mathbb{Q}_\ell$. For $i \leqslant n-2$ the ℓ-adic sheaf $\mathcal{P}^i$ is constant of fibre $P^i(X)$. To sum up, for every Lefschetz fibration ρ one has a Lefschetz theory on the sheaves $\nu_*\nu^*\mathcal{R}^i$ (which are independent of ν chosen as above), whose primitive pieces are the above defined $\mathcal{P}^i$. This Lefschetz decomposition turns into one for the ℓ-adic sheaves $\mathcal{R}^i$ if **(A)** holds for ρ.

Corollary 4.5. *Notations as above. The following statements hold:*

1. *The following isomorphisms hold:*

$$L^{n-1-i} : R^i\rho_*\mathbb{Q}_\ell \to R^{2n-2-i}\rho_*\mathbb{Q}_\ell$$

   for $i \neq n-2, n$, and for all i if **(A)** *holds for ρ.*
2. *Let $\mathcal{P}^i = \mathcal{P}^i_\rho = \operatorname{Ker} L^{n-i} \subset R^i\rho_*\mathbb{Q}_\ell$. Then $\mathcal{P}^i_\rho$ is constant of fibre $P^i(X)$ if $i \leqslant n-2$ and $\mathcal{P}^{n-1}_\rho = \mathcal{E}^{n-1} \oplus P^{n-1}(X)_{\mathbb{P}^1}$, where $\mathcal{E}^{n-1} = \nu_*\nu^*\mathcal{E}^{n-1}$ for all ν: $\mathcal{U} \subset \mathbb{P}^1$ within the smooth locus of ρ; moreover $\mathcal{E}^{n-1}_{\bar{t}} = V(X_t)$ for $\bar{t} = t \in \mathcal{U}(k)$ (as $k = \bar{k}$) and also for $\bar{t} \to t = \eta \in \mathcal{U}$ a generic geometric point.*
3. *Let $0 \leqslant \epsilon \leqslant 2$, $0 \leqslant i \leqslant 2n-2$. The pairings*

$$R^i\rho_*\mathbb{Q}_\ell \times R^{2n-2-i}\rho_*\mathbb{Q}_\ell \to R^{2n-2}\rho_*\mathbb{Q}_\ell \simeq \mathbb{Q}_\ell$$

   and

$$L^{n-1-i} \bullet \cup \bullet : \mathcal{P}^i_\rho \times \mathcal{P}^i_\rho \to \mathbb{Q}_\ell \tag{6}$$

   induce perfect pairings

$$H^\epsilon\big(R^i\rho_*\mathbb{Q}_\ell\big) \otimes H^{2-\epsilon}\big(R^{2n-2-i}\rho_*\mathbb{Q}_\ell\big) \to H^2\big(\mathbb{P}^1, \mathbb{Q}_\ell\big)$$

   for $(\epsilon, i) \neq (0, n), (2, n-2)$ if **(A)** *does not hold and for all ϵ, i if* **(A)** *holds; the induced pairings*

$$H^\epsilon\big(\mathcal{P}^i_\rho\big) \otimes H^{2-\epsilon}\big(\mathcal{P}^i_\rho\big) \to \mathbb{Q}_\ell \tag{7}$$

   for $0 \leqslant i \leqslant n-1$ are also perfect. All these pairings agree with the ones resulting from Theorem 4.3*; for instance, the pairing given by $a \otimes b \mapsto \langle L^{n-1-i}a, b\rangle_{\tilde{X}}$ in $Gr^\bullet_{F_\rho} H^*(\tilde{X})$ equals the one in* (6).

One has $\dim H^0(\mathcal{P}^i_\rho) = \dim H^2(\mathcal{P}^i_\rho)$ *for* $0 \leqslant i \leqslant n-1$.

4. *The Lefschetz isomorphisms on sheaves given in Lemma* 4.1 *translate also into their cohomology groups, namely* $L^{n-1-i}\colon\ H^\epsilon(\mathcal{R}^i) \cong H^\epsilon(\mathcal{R}^{2n-2-i})$ *for* $i \leqslant n-3$ *and for* $\epsilon = 1, 2$ *also* $H^\epsilon(\mathcal{R}^{n-2}) \cong H^\epsilon(\mathcal{R}^n)$*; in particular, for* $0 \leqslant \epsilon \leqslant 2$, $0 \leqslant i \leqslant n-1$ *we have*

$$H^\epsilon\big(\mathcal{P}^i_\rho\big) = \ker\big(L^{n-i} : H^\epsilon\big(\mathcal{R}^i\big) \to H^\epsilon\big(\mathcal{R}^{2n-i}\big)\big).$$

5. $\dim H^0(\mathcal{R}^i) = \dim H^2(\mathcal{R}^i) = b_i(X)$ *for* $i \leqslant n-1$, *and* $\dim H^2(\mathcal{R}^n) = b_{n-2}(X)$. *As a result,* $\dim H^0(\mathcal{P}^i_\rho) = \dim H^2(\mathcal{P}^i_\rho) = \dim P^i(X)$ *for all* $i \leqslant n-1$. *The left kernel of the pairing*

$$H^0\big(\mathcal{R}^n\big) \otimes H^2\big(\mathcal{R}^{n-2}\big) \to H^2\big(\mathcal{R}^{2n-2}\big) \cong \mathbb{Q}_\ell$$

is $H^0(\mathcal{K})$*, which is zero iff* **(A)** *holds.*

Proof. The result follows from Lemma 4.2, Theorem 4.3, Deligne [4, 2.8 and 2.12], and Katz [6, XVIII Lemmas 5.4, 5.5, 5.6.9 and proof of Th. 5.6.8].

Let us check the last assertion for $i = n-1$: the morphism $k_* = \iota_* \oplus -h^*|H^{n-1}(Y)$ has kernel $V(Y) = \ker \iota_*$. Therefore $\dim H^2(\mathcal{R}^{n-1}) = b_{n-1}(X) = b_{n-1}(Y) - \dim V(Y)$. The equality $\mathcal{R}^i = \mathcal{P}^i_\rho \oplus L\mathcal{R}^{i-2}$ yields $\dim H^0(\mathcal{P}^i_\rho) = \dim P^i(X)$. (Alternatively, use [6, XVIII Th. 5.6].) For the last assertion of 5 see Lemma 4.2. □

Now let us go back to the computations of Proposition 3.3.

Lemma 4.6. *Let* $x \oplus y \in H^i(\tilde{X})$*, and let* $r \in \mathbb{N}$. *Then* $\xi_{\tilde{X}} - m\cdot f^*(\xi_X) = \rho^*([t]) = k_*(1_{H^*(Y)}) \in F^2_\rho$. *Thus the expression*

$$\big(L^r_{\tilde{X}} - m^r L^r\big)(x \oplus y) = L^r_{\tilde{X}}(x \oplus y) - m^r\big(L^r x \oplus L^r_\Delta y\big) = r\cdot m^{r-1} k_* L^{r-1}_Y(\iota^* x + h_* y) \tag{8}$$

belongs to F^2_ρ.

Proof. By (5) we have $L^s(x \oplus y) = f^*\xi^s_X \wedge (x \oplus y) = L^s x \oplus L^s_\Delta y$. On the other hand, if $Y = X_t$ is a smooth geometric fibre, then $\rho^*([t]) = k_*(1_Y) = \xi_X \oplus -1_\Delta \in H^2(\tilde{X})$. We have $\rho^*(t) \wedge (x \oplus 0) = Lx \oplus -j^* x = k_*(\iota^* x)$ and $\rho^*(t) \wedge (0 \oplus y) = j_* y \oplus -L_\Delta y = k_*(h_* y)$, whence $\rho^*(t) \wedge (x \oplus y) = k_*(\iota^* x + h_* y) = (Lx + j_* y) \oplus -(j^* x + L_\Delta y)$. Finally

$$\begin{aligned} r\cdot m^{r-1} L^r x + L^{r-1} j_* y \oplus -L^{r-1}_\Delta(j^* x + L_\Delta y) &= L^r_{\tilde{X}}(x \oplus y) - m^r\big(L^r x \oplus L^r_\Delta y\big) \\ &= r\cdot m^{r-1} k_* L^{r-1}_Y(\iota^* x + h_* y) \end{aligned}$$

as desired. □

Corollary 4.7. *Notations and assumptions being as above,*

$$L^{n-i}_{\tilde{X}}\big(P^i(X) \oplus 0\big) = L^{n-i} P^i(X) \oplus 0 = k_* L^{n-i-1}_Y \iota^* P^i(X) \subset F^2_\rho$$

and $P^i(\tilde{X}) \supset P^i(X) \oplus 0$. *One has* $L^r_{\tilde{X}} k_*(y) = m^r k_*(L^r_Y y) = m^r L^r k_* y$ *for all* $r \geqslant 0$.

Proof. By [16, 1.4.7], $h^* L_Y^{n-1-i} P^i(Y) = 0$ and

$$k_* : L_Y^{n-i-1} \iota^* P^i(X) \to L^{n-i} P^i(X) \oplus 0$$

is an isomorphism. Let us prove the inclusion $P^i(X) \oplus 0 \subset P^i(\tilde{X})$. By formula (8), it suffices to check that

$$(n-i) m^{n-i} k_* L_Y^{n-i-1} P^i(Y) \subset L^{n-i} P^i(X) \oplus 0,$$

but this inclusion is clear. We have seen that the image of $P^i(X) \oplus 0$ via the Lefschetz isomorphism is precisely $L^{n-i} P^i(X) \oplus 0$, thus establishing the result. □

Remark 4.8. By Lemma 4.6, the operator $L_{\tilde{X}} - m \cdot L$ vanishes on $Gr_F^* H^*(\tilde{X})$. It is not difficult to check that it also vanishes on the sheaves $\mathcal{R}^i$ as well.

Corollary 4.9. *Let $i \leqslant n-1$. The map L^{n-i} and the Lefschetz isomorphism $L_{\tilde{X}}^{n-i}$ yield isomorphisms*

$$\bigl(P^i(X) \oplus 0\bigr) \oplus k_* H^{i-2}(Y) \xrightarrow{\sim} k_* H^{2n-2-i}(Y).$$

The subspace $L_{\tilde{X}}^j P^i(X) \oplus 0$ is linearly disjoint with F_ρ^1 for $j < n-i$, and $L^{n-i} P^i(X) \oplus 0 \subset F_\rho^2$.

Proof. The first assertion follows from Corollary 4.7 and Corollary 4.5(5). The second assertion follows from the first. □

Corollary 4.10. *The natural map $P^i(X) \oplus 0 \to H^0(\mathcal{R}^i)$ of Theorem* 4.3 *induces an isomorphism*

$$P^i(X) \oplus 0 \cong H^0\bigl(\mathcal{P}_\rho^i\bigr)$$

for $0 \leqslant i \leqslant n-1$. The map $\rho^*(t) \wedge \bullet$ yields an isomorphism between $H^0(\mathcal{P}_\rho^i)$ and $H^2(\mathcal{P}_\rho^i)$. As a result, $H^2(\mathcal{P}_\rho^i) = L P^i(X) \oplus P^{i-2}(\Delta) \cap F_\rho^2 H^{i+2}(\tilde{X}) = k_* P^i(Y)$ for $i \leqslant n-1$.

Likewise, the map $H^i(X) \oplus 0 \to H^0(\mathcal{R}^i)$ is an isomorphism for $i \leqslant n-1$. One has $H^0(\mathcal{E}^{n-1}) = H^2(\mathcal{E}^{n-1}) = 0$.

Proof.

1. The dimensions are equal, and $L^{n-i}(P^i(X) \oplus 0) \subset F_\rho^2$, hence the map

$$P^i(X) \oplus 0 \to H^0\bigl(\mathcal{R}^i\bigr)$$

   induces an isomorphism onto $H^0(\mathcal{P}_\rho^i)$.
2. The class $[\rho^*(t)] \in \rho^* H^2(\mathbb{P}^1)$, hence $\rho^*(t) \wedge \bullet$ induces a map $H^0(\mathcal{P}_\rho^i) \to H^2(\mathcal{P}_\rho^i)$, which reads as follows:

$$\rho^*(t) \wedge (x \oplus 0) = k_* k^*(x \oplus 0) = k_* \iota^* x.$$

Therefore its image is $k_*\iota^* P^i(X)$, whose dimension agrees with $\dim H^2(\mathcal{P}^i_\rho)$ ($H^2(\mathcal{P}^i)$ is precisely the kernel of L^{n-1-i} within $H^2(\mathcal{R}^i) = k_* H^i(Y)$, which agrees with $k_*\iota^* P^i(X)$). The first assertion is thus proven.

The second assertion is straightforward: one has $H^i(X) \oplus 0 \cap k_* H^{i-2}(Y) = 0$, which means that $H^i(X) \oplus 0$ maps isomorphically onto its image in $F^0/F^1 H^i(\tilde{X})$ for $i \leqslant n-1$. Equality of dimensions yields the desired isomorphism. For the third assertion, by Corollary 4.5 we may split $\mathcal{R}^{n-1} = H^{n-1}(X)_{\mathbb{P}^1} \oplus \mathcal{E}^{n-1}$, and applying H^0 or H^2 yields $H^0(\mathcal{E}^{n-1}) = H^2(\mathcal{E}^{n-1}) = 0$. The corollary is thus established. □

4.1. Absolute and relative correspondences

Let $p: M \to B$ be a smooth projective morphism onto a smooth algebraic variety B; denote the dimension of M by n. If u is a codimension-$(r - \dim B)$ cycle on $M \times_B M$, the degree of u as a relative correspondence is defined to be r, i.e. the same as that of u as a correspondence of M; thus the cycle Δ_M is a relative correspondence of degree 0 (see Fulton [8, Chs. 10, 16], or Künnemann [14]). If $u, v \in CH^{n-1+*}(M \times_B M)$, the composition of u, v relative to B is defined to be

$$v \circ_B u := p^B_{13*}\big(p^{B*}_{12}(u) \bullet p^{B*}_{23}(v)\big),$$

where $p^B_{ij}: M \times_B M \times_B M \to M \times_B M$ are the canonical projections. $\circ_B$ endows $CH^{n-1+*}(M \times_B M)$ with a ring structure, and the usual properties hold. The fact is, if in addition M and B are projective this relative product is related to the usual composition of correspondences from M to M, as we will show. However, we need to extend this theory so that the case of Lefschetz fibrations will be included: the heart of this section is Proposition 4.12, where every identity holds modulo rational equivalence. The upshot is Proposition 4.15, where we show that relative correspondences do preserve a certain filtration in the case when B is a curve, which in the case of Lefschetz fibrations satisfying **(A)** coincides with the Leray filtration of ρ.

In the case when f is smooth, we obtain the following proposition:

Proposition 4.11. *Let $f: M \to B$ be a smooth projective fibration (i.e. with connected fibres) over a smooth proper base B. Every correspondence u supported on $M \times_B M$ preserves the Leray filtration of f.*

Proof. Denote by $p'_i: M \times_B M \to M$ the natural (relative) projections, and by p_i those corresponding to the fibre product; denote by $\varphi: M \times_B M \to B$ the structure morphism. Let $j: M \times_B M \hookrightarrow M \times M$ denote the canonical inclusion (so that $p'_i = p_i \circ j$). For a correspondence $u \in A^*(M \times_B M)$ we have

$$p_{2*}\big(j_*(u) p_1^*(x)\big) = p'_{2*}\big(u p_1'^*(x)\big), \tag{9}$$

by the projection formula $j_*(u)p_1^*(x) = j_*(u j^* p_1^*(x))$. Now, functoriality of the Leray spectral sequence implies $p_1'^* F^r_f \subset F^r_\varphi$, which is clearly annihilated by $p_2'^* F_f^{2b-r+1}$ where $b = \dim B$. This shows that $j_*(u) F^r_f$ is annihilated by F_f^{2b-r+1}, which amounts to saying $j_*(u) F^r_f \subset F^r_f$ by the degeneration of the Leray spectral sequence on f at E_2, thus establishing the proposition. □

When dealing with Lefschetz fibrations in general, things are slightly different.

Let $f : M \to B$ be a flat projective map of smooth quasiprojective varieties over a field. Let $\mathcal{R} := CH_*(M \times_B M)$ (denoted $\mathcal{R}_f$ when disambiguation is necessary). We will operate in this slightly more general setting, which includes the case of Lefschetz fibrations.

Note that the natural inclusion $M^{\times_B n} \hookrightarrow M^n$ is a regular embedding, which follows from pulling back the regular embedding $(id_B, \ldots, id_B) : B \hookrightarrow B^n$ to the smooth variety M^n via f^n (use [12, Th. 8.21.A(c)]).

Denote by q_{ij} the natural relative projections $M \times_B M \times_B M \to M \times_B M$. We may define the following composition law on $\mathcal{R}$:

$$v \circ_B u := q_{13*}(q_{12}, q_{23})^!(u \times v).$$

This composition law is well-defined since the map

$$(q_{12}, q_{23}) : M \times_B M \times_B M \to M \times_B M \times M \times_B M$$

is a regular imbedding, as follows from repeated application of [12, Th. 8.21.A(c)]. Indeed, (q_{12}, q_{23}) is a base change from the regular embedding $(p_{12}, p_{23}) : M^3 \hookrightarrow M^2 \times M^2$, and the local equations defining $(p_{12}, p_{23})(M^3)$ within M^4 around a point and those for $M \times_B M \times M \times_B M \subset M^4$ (around the same point) together form a regular sequence, which yields (q_{12}, q_{23}) a regular embedding.

Proposition 4.12. *Notations and assumptions of this subsection are kept.*

1. *The composition law $\circ_B$ is associative.*
2. *The class $[\Delta_M]$ is the unit element in $\mathcal{R}$, which makes $(\mathcal{R}, +, \circ_B)$ into a unital ring.*
3. *Assume that B, hence M, is projective. The natural inclusion $j : M \times_B M \hookrightarrow M \times M$ satisfies $j_*(v \circ_B u) = j_* v \circ j_* u$, where $\circ$ is the usual composition of self-correspondences of M. In other words, j_* induces a homomorphism of rings, which translates into $\lambda : \mathcal{R} \otimes \mathbb{Q} \to \mathrm{End}(h_{rat}(M))$.*

Proof. The proof of 1 goes along the lines of [8, Prop. 16.1.1.(a)], taking due care in replacing every instance of cup-product by an operator of the type $i^!$ for i a suitable regular embedding; the pullback diagrams we deal with are thus of the type found in [8, Ch. 6], esp. [8, Th. 6.2] in which the "horizontal" arrows are regular embeddings:

$$\gamma \circ_B (\beta \circ_B \alpha) = q^{134}_{14*}\big(q^{134}_{13}, q^{134}_{34}\big)^! \big(q^{123}_{13*}\big(\big(q^{123}_{13}, q^{123}_{23}\big)^!(\alpha \times \beta)\big) \times \gamma\big).$$

Now, using the corresponding Cartesian diagram we have

$$\big(q^{134}_{13}, q^{134}_{34}\big)^! \big(q^{123}_{13} \times id_{34}\big)_* = q_{134*}(q_{123}, q_{34})^!,$$

which by the above implies

$$\gamma \circ_B (\beta \circ_B \alpha) = q_{14*}(q_{123}, q_{34})^! \big(\big(q^{123}_{12}, q^{123}_{23}\big)^!(\alpha \times \beta) \times \gamma\big) = q_{14*}(q_{12}, q_{23}, q_{34})^!(\alpha \times \beta \times \gamma),$$

and this in turn equals $(\gamma \circ_B \beta) \circ_B \alpha$ by a similar procedure. Associativity is thus established.

Now we show 2. Let us check that $q_{13*}(q_{12},q_{23})^!(\Delta_M\times\alpha)=\alpha$. For this we consider the following Cartesian diagram (Cartesian property is easily checked locally):

$$\begin{tikzcd} M\times_B M \arrow[r, "\Delta_{M/B}\times_B id_M"] \arrow[d, "(q_1^{12}, id_{M\times_B M})"'] & M\times_B M\times_B M \arrow[d, "(q_{12},q_{23})"] \\ M\times M\times_B M \arrow[r, "\Delta_{M/B}\times id_{M\times_B M}"] & (M\times_B M)^2. \end{tikzcd} \tag{10}$$

The above diagram yields

$$\begin{aligned}[][\Delta_M]\circ_B\alpha &= q_{13*}(q_{12},q_{23})^!(\Delta_M\times id_{M\times_B M})_*\big([M]\times\alpha\big)\\ &= q_{13*}(\Delta_{M/B}\times_B id_M)_*\big(q_1^{12}, id_{M\times_B M}\big)^!(M\times\alpha)=\alpha.\end{aligned}$$

The last equality holds by the following: one has $M\times\alpha=proj_2^*\alpha$ where

$$proj_2: M\times(M\times_B M)\to M\times_B M$$

is the natural projection onto the second factor. Now, both arrows $(q_1^{12}, id_{M\times_B M})$ and $proj_2$ are obtained by base change ($M\times_B M\hookrightarrow M\times M$) from the maps $M^2\overset{i}{\hookrightarrow} M\times M^2\overset{p}{\twoheadrightarrow} M^2$ where $i(x,y)=(x,x,y)$ and $p(x,y,z)=(y,z)$, we are in the hypotheses of [8, Prop. 6.5.(b)], which means that $i^!p^*=(pi)^!$, which is id in our case, and so with any base change as is our case, by [8, Prop. 6.5.(b)] Hence the final result $[\Delta_M]\circ_B\alpha=\alpha$. The identity $\alpha\circ_B[\Delta_M]=\alpha$ is completely analogous. Part 2 is thus settled.

We now proceed to show part 3 under projectiveness assumptions on B, hence on M. We wish to see that $j_*(v\circ_B u)=j_*(v)\circ j_*(u)$.

Developing both sides one has:

$$j_*q_{13*}(q_{12},q_{23})^!(u\times v)=p_{13*}\big(p_{12}^*(j_*u)\bullet p_{23}^*(j_*v)\big).$$

Since $j_*q_{13*}=p_{13*}inc_*$, it suffices to show

$$inc_*(q_{12},q_{23})^!(u\times v)=p_{12}^*(j_*u)\bullet p_{23}^*(j_*v)=(p_{12},p_{23})^!(j\times j)_*(u\times v). \tag{11}$$

The following diagram is Cartesian:

$$\begin{tikzcd} M\times_B M\times_B M \arrow[r, "inc"] \arrow[d, "(q_{12},q_{23})"'] & M^3 \arrow[d, "(p_{12},p_{23})"] \\ M\times_B M\times M\times_B M \arrow[r, "j\times j"] & M^2. \end{tikzcd}$$

We remark that all morphisms in the above diagram are regular imbeddings. By [8, Th. 6.2] we have

$$inc_*\big(p_{12}^B, p_{23}^B\big)^!=(p_{12},p_{23})^!(j\times j)_* \tag{12}$$

on Chow groups, which proves (11) and thus settles 3. The proposition is now established. □

Now consider M, B as above. We have the inclusion maps $\iota_t^n : M_t^n \hookrightarrow M^{\times_B n}$ which result from specialising the structure morphism $M^{\times_B n} \to B$ to $\{t\} \hookrightarrow B$. Flatness of f at t yields ι_t^n a regular imbedding [12, Ex. 10.9]. As usual, for Z an algebraic cycle in $M^{\times_B n}$ we denote $Z_t := \iota_t^{n!} Z$.

Lemma 4.13. *Let $u, v \in \mathcal{R}$. Assume that $t \in B$ is a smooth point of f. Then*

$$(v \circ_B u)_t = v_t \circ u_t.$$

If in addition B and so M are projective then

$$j_* u \circ \iota_{t*} = \iota_{t*} \circ u_t. \tag{13}$$

Proof. For the first assertion, one may suppose M smooth over B, by splitting $\{t\} \subset B^0 \subset B$ with B^0 the smooth locus of f. One may then write $v \circ_B u = q_{13*}(q_{12}, q_{23})_B^*(u \times v)$, where $(q_{12}, q_{23})_B : M^{\times_B 3} \hookrightarrow M^{\times_B 4}$ denotes the usual embedding (relative to B). Denote all natural projections on M_t with p', e.g. $p'_{12} : M_t^3 \to M_t^2$. The basic pullback diagrams show $\iota_t^{2!} q_{13*} = p'_{13*} \iota_t^{3!}$; commutation and contravariance for non-singular varieties yield $\iota_t^{3*}(q_{12}, q_{23})_B^* = (p'_{12}, p'_{23})^* \iota_t^{4*}$, and so $(v \circ_B u)_t = p'_{13*}(p_{12}, p_{13})^*[\iota_t^{4*}(u \times_B v)] = v_t \circ u_t$ as desired.

The second assertion may be written in the following fashion: $(\iota_t \times 1)^* j_*(u) = (1 \times \iota_t)_* \iota_t^{2!}(u)$, and follows once more from [8, Th. 6.2]. $\square$

Proposition 4.14. *Let $f : M \to B$ be as in this subsection. Assume f to be generically smooth (as is always the case in characteristic zero).*

1. *If $B_2 \subset B_1$ are open subsets of B, we have natural restriction maps $\mathcal{R}_f \twoheadrightarrow \mathcal{R}_{f_1}$ and $\mathcal{R}_{f_1} \twoheadrightarrow \mathcal{R}_{f_2}$ which are ring homomorphisms—here f_i denote the corresponding restrictions of f to $f^{-1}(B_i) \to B_i$.*
2. *Let η is the generic point of B; let $\mathcal{M}$ denote the generic fibre of f. We have a natural epimorphism of rings $\bar{r}_\eta : \mathcal{R}_f \otimes \mathbb{Q} \twoheadrightarrow \operatorname{End} h_{rat}(\mathcal{M})$.*
3. *Under the assumptions of 2, define $\mathcal{I}^0 \subset \mathcal{R} \otimes \mathbb{Q}$ to be the subset of cycle classes in $\mathcal{R} \otimes \mathbb{Q}$ which are sent to homologically trivial self-correspondences of $\mathcal{M}$ by the restriction map above. Then: $\mathcal{I}^0$ is an ideal of $\mathcal{R} \otimes \mathbb{Q}$.*

Proof. Property 1 is clear. The kernel of $\mathcal{R}_{f_1} \twoheadrightarrow \mathcal{R}_{f_2}$ is precisely the image in $\mathcal{R}_{f_1}$ of

$$CH_*\big(f^{-1}(B_1 \setminus B_2) \times_{B_1 \setminus B_2} f^{-1}(B_1 \setminus B_2)\big).$$

Part 2 follows from Lemma 4.13.

To establish 3, one may consider the ring homomorphism

$$\bar{r}_\eta : \mathcal{R}_f \otimes \mathbb{Q} \twoheadrightarrow \operatorname{End} h_{rat}(\mathcal{M}) \twoheadrightarrow \operatorname{End} h(\mathcal{M}),$$

whose kernel turns out to be $\mathcal{I}^0$. Part 3 is now complete. $\square$

Proposition 4.15. *(Assume $k = \bar{k}$.) Let $f : M \to B$ be a generically smooth morphism from a smooth projective variety M onto a smooth proper curve B. Let t denote a smooth point of B. One has the following filtration on $H^*(M)$:*

$$\tilde{F}^0 = H^*(M) \supset \tilde{F}^1 = \operatorname{Ker} \iota_t^* \supset \tilde{F}^2 = \operatorname{Im} \iota_{t*}.$$

The definition is independent of the point t chosen (one may even choose a geometric generic point of B, and extend scalars to $\overline{k(B)}$). Every self-correspondence of M supported on $M \times_B M$ preserves the filtration $\tilde{F}$.

Let $\rho : \tilde{X} \to \mathbb{P}^1$ be an arbitrary Lefschetz fibration. The correspondences supported on $D = \tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ preserve the filtration $\tilde{F}^*$, which coincides with the Leray filtration if **(A)** *holds. Let $u \in CH_*(\tilde{X} \times_{\mathbb{P}^1} \tilde{X})$; denote the natural inclusion by $j : \tilde{X} \times_{\mathbb{P}^1} \tilde{X} \hookrightarrow \tilde{X} \times \tilde{X}$. We have $[j_*(u)]F_\rho^i \subset F_\rho^i$ for $i = 0, 1, 2$ (for then, and only then, the filtration $\tilde{F}$ and the Leray filtration are equal). Moreover, if u is supported on a finite set of fibres of the structure morphism $D = \tilde{X} \times_{\mathbb{P}^1} \tilde{X} \to \mathbb{P}^1$, then $[j_*u]|\tilde{F}^2 = 0$, $[j_*u]H^*(\tilde{X}) \subset \operatorname{Ker} k^*$; if furthermore* **(A)** *holds, then such a correspondence acts on the Leray filtration of ρ sending F^i to F^{i+1} for $i = 0, 1, 2$.*

Proof. First we remark that the filtration $\tilde{F}$ is independent of the point chosen. Denote by B^0 the smooth locus of f, $M^0 := f^{-1}(B^0)$ and $f^0 : M^0 \to B^0$. Define the map $\Phi : M^0 \hookrightarrow M \times B^0$ by $\Phi(x) := (x, f(x))$ which is a morphism of B^0-schemes if we take the second projection on the target as the structure morphism; fibrewise we have ι_t for $t \in B^0$. Taking $t = \eta$ the generic fibre of B, Φ reduces to the natural inclusion $\mathcal{M} \hookrightarrow M$ where $\mathcal{M}$ is the generic fibre of f. Φ naturally induces morphisms of local systems $H^i(M)_{B^0} \to R^i f_*^0 \mathbb{Q}_\ell$, and smooth specialisation [18, Cor. VI.4.2] yields independence of $t \in B^0$ (including η) for $\operatorname{Ker} \iota_t^*$.

Now, by (13), any correspondence v supported on $M \times_B M$ preserves $\tilde{F}$. Indeed, $[j_*v] \circ \iota_{t*} = \iota_{t*} \circ v_t$ clearly implies that $\tilde{F}^2$ is preserved by $[j_*v]$. Transposing this identity yields the preservation of $\tilde{F}^1$ under the action of any correspondence of this type.

Let us consider the case of a Lefschetz fibration ρ. In this case, $\tilde{F}^2 = F^2$, and $\tilde{F}^1 H^i(\tilde{X}) = F^1 H^i(\tilde{X}) = F^2 H^i(\tilde{X})$ for $i \neq n$, and $\tilde{F}^1 H^n(\tilde{X}) \supset F^1 H^n(\tilde{X})$. If in addition **(A)** holds, then the Leray filtration and $\tilde{F}^*$ coincide, so $F^* = F_\rho^*$ is then preserved by all correspondences supported on D (relative correspondences). Let v be a relative correspondence supported on $\bigcup_{1 \leqslant i \leqslant m_0} M_{t_i} \times M_{t_i}$; taking ι_t for $t \neq t_i$ smooth, one easily sees $v\tilde{F}^2 = 0$ and the dual statement $\iota_t^* w = 0$. If **(A)** holds for ρ, then $w = j_*v$ with v_η homologically trivial on $\mathcal{Y} \times \mathcal{Y}$, and by Remark 4.19 we get $wF^i \subset F^{i+1}$. The proof is now complete. □

Proposition 4.16. *M, B and f being as in Proposition* 4.15*, any correspondence u supported on $(f \times f)^{-1}(D)$ with D a 1-dimensional subscheme of $B \times B$ containing no component of the form $B \times s$ with s a singular value of f, preserves the filtration $\tilde{F}^*$. For arbitrary $D \neq B \times B$, $u\tilde{F}^2 \subset \tilde{F}^1$.*

Proof. For the first part, choose t a point of B such that $D \cap t \times \mathbb{P}^1$ is finite and lies within the smooth locus of $f \times f$, then $u \circ \iota_{t*} = (\iota_t \times 1)^* u$ is supported on $(f \times f)^*(D \cap t \times \mathbb{P}^1)$, and so $u \circ \iota_{t*} = \sum \iota_{t_i *} w_i$, with w_i supported on $M \times M_{t_i}$. This shows that u preserves $\tilde{F}^2$. Similarly one sees that ${}^t u$ preserves $\tilde{F}^2$, i.e. u preserves $\tilde{F}^1$. The last part is left to the reader. □

Proposition 4.17. *(See Katz [6, Exp. XVIII Th. 5.7].) Let $\rho : \tilde{X} \to \mathbb{P}^1$ be an arbitrary Lefschetz fibration. Then $\tilde{F}^1 H^i(\tilde{X}) = \tilde{F}^2 H^i(\tilde{X}) = F^2_\rho H^i(\tilde{X})$ for $i \neq n$, and $\tilde{F}^1 H^n(\tilde{X}) = (P^n(X) \oplus V(\Delta)) \oplus \tilde{F}^2 H^i(\tilde{X})$. Also*

$$H^0(\mathcal{K}) \oplus H^1(\mathcal{R}^{n-1}) \cong P^n(X) \oplus V(\Delta).$$

Proof. See Theorem 4.3 for the first assertion. The second assertion is elementary. For the last part, see the reference provided. □

4.2. Action on the Leray spectral sequence

(Again we assume k algebraically closed.) Let u be a correspondence of degree r supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$. Then u induces a correspondence u_t of degree r on X_t for each $t \in B$, where $\nu : B \to \mathbb{P}^1$ is the smooth locus of ρ as above. If **(A)** holds, then u defines a homomorphism of ℓ-adic sheaves for $0 \leqslant j \leqslant n-1$:

$$u : \nu_* \nu^* R^j \rho_* \mathbb{Q}_\ell = R^j \rho_* \mathbb{Q}_\ell \to \nu_* \nu^* R^{j+2r} \rho_* \mathbb{Q}_\ell = R^{j+2r} \rho_* \mathbb{Q}_\ell \tag{14}$$

(using **(A)**), which in turn yields $\mathbb{Q}_\ell$-linear maps

$$H^i(R^j \rho_* \mathbb{Q}_\ell) \to H^i(R^{j+2r} \rho_* \mathbb{Q}_\ell). \tag{15}$$

These maps agree with those induced on $Gr^*_{F_\rho}$ by $j_* u$ in Proposition 4.15 (more on this later), and so do the respective composition laws.

Remark 4.18. Assume that **(A)** holds for ρ. Morphisms induced in (14) and (15) depend only on the class of u in $H^*(\mathcal{Y} \times \mathcal{Y})$ (this is the case if the degree of u is not 0 if **(A)** does not hold). Indeed, denote the generic point of $\mathbb{P}^1$ by η, and the image of u in $A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$ by u_η or $[u]_{\mathcal{Y}}$; suppose that $u'_\eta - u''_\eta | H^j(\mathcal{Y}) = 0$ for all j. Then for a sufficiently small neighbourhood $\nu_1 : \mathcal{U}_1 \subset \mathbb{P}^1$ of η one has $0 = u' - u'' | \nu_{1*} \nu_1^* R^j \rho_* \mathbb{Q}_\ell = R^j \rho_* \mathbb{Q}_\ell$ for all j, hence $u' - u''$ induces 0 on $Gr^*_{F_\rho} H^*(\tilde{X})$. Here we used [7, I.12.10, I.12.13] (see also [11]) and the base change theorems in étale cohomology [10]; [7, I.6, I.7].

Remark 4.19. Let ρ be an arbitrary Lefschetz fibration. Let $v \in A^*(\mathcal{Y} \times \mathcal{Y})$, and let u be a lifting of v. We can identify $\tilde{F}^0 / \tilde{F}^1 = H^*(\mathcal{Y})^G$ and $\tilde{F}^2 = F^2 = H^{*-2}(\mathcal{Y})_G$ where $G = Gal(k(\mathbb{P}^1)^{sep} / k(\mathbb{P}^1))$ acts through its quotient $G' = \pi_1(B, \bar{\eta})$, B being the smooth locus of ρ: for the first identification see Theorem 4.3. The second follows from the chain of identities $F^2 H^*(\tilde{X}) = H^2(\mathcal{R}^{*-2}_\rho) = H^2(\nu_* \nu^* \mathcal{R}^{*-2}_\rho) = H^{*-2}(\mathcal{Y})_G$ [11, I]. Then the action of u on $\tilde{F}^0 / \tilde{F}^1$ (resp. F^2) corresponds to the action of v on $H^*(\mathcal{Y})^G$ (resp. $H^{*-2}(\mathcal{Y})_G$).

Definition. Assume **(A)** as above. Let $\mathcal{A} \subset A^{n+*}(\tilde{X} \times \tilde{X})$ denote the subring (see Proposition 4.12) of homological correspondences supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$. Let $\mathcal{J}$ be the ideal of $\mathcal{A}$ consisting of the elements u such that $u F^i_\rho \subset F^{i+1}_\rho$ for $i = 0, 1, 2$ (i.e. those inducing 0 on $Gr^*_{F_\rho} H^*(\tilde{X})$). Let $\mathcal{I}$ be the ideal (see Proposition 4.14) of $\mathcal{A}$ consisting of the $u \in \mathcal{A}$ such that $u = [j_* v]$ with v an algebraic cycle on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ (with $\mathbb{Q}$-coefficients) inducing a homologically trivial class on $H^*(\mathcal{Y} \times \mathcal{Y})$ (which is the homomorphic image of $\mathcal{I}^0$ via the ring epimorphism

$\bar{r}_\eta : \mathcal{R}_\rho \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A}$). We denote by $\mathcal{J}' \subset \mathcal{A}$ the subspace of all classes $w = [j_* w_0] \in \mathcal{A}$ satisfying $[w_0]_{\mathcal{Y}} H^*(\mathcal{Y}) \subset V(\mathcal{Y})$.

The following proposition sharpens Remark 4.18 above.

Proposition 4.20. *Notations and assumptions as above (in particular we assume* **(A)***). The subspaces $\mathcal{I}, \mathcal{J}, \mathcal{J}'$ of $\mathcal{A}$ are all ideals, and satisfy $\mathcal{I} \subset \mathcal{J}, \mathcal{I}^{\circ 3} = \mathcal{J}^{\circ 3} = 0$. Let w_0 be an algebraic cycle supported on D, representing the correspondence $w \in A^{n+r}(\tilde{X} \times \tilde{X})$ of degree r. Suppose that $r \neq 0$; then $w_0 \in \mathcal{I}$ if and only if $w \in \mathcal{J}'$. In general, the following statements hold.*

- (i) *$\mathcal{I}_r = \mathcal{J}_r = \mathcal{J}'_r$ for $r \neq 0$. $\mathcal{J} \subset \mathcal{J}'$, and $\mathcal{J}'$ is an ideal of $\mathcal{A}$.*
- (ii) *Suppose that $H^1(R^{n-1}\rho_* \mathbb{Q}_\ell) = 0$. Then $\mathcal{J} = \mathcal{J}'$.*
- (iii) *Assume that n is even or* $\operatorname{char} k \neq 2$. *If $H^1(R^{n-1}\rho_* \mathbb{Q}_\ell) \neq 0$, then $\mathcal{I} = \mathcal{J}$. Moreover, if $B(X)$ holds one has $\mathcal{J} \subsetneq \mathcal{J}'$.*
- (iv) *If n is even or* $\operatorname{char} k \neq 2$, *then there exists $d_0 \in \mathbb{N}$ such that, for every $d \geqslant d_0$, every Lefschetz fibration of degree-d hypersurfaces satisfies* **(A)** *and* (iii).
- (v) *If n is even or* $\operatorname{char} k \neq 2$, *and $H^1(R^{n-1}\rho_* \mathbb{Q}_\ell) \neq 0$, then any Chow class u supported on D such that $[j_* u] = 0$ satisfies $[u]_{\mathcal{Y}} = 0$.*

Proof. $\mathcal{I}$ and $\mathcal{J}$ are clearly ideals of $\mathcal{A}$. The nilpotence assertion for $\mathcal{I}, \mathcal{J}$ is clear—the inclusion $\mathcal{I} \subset \mathcal{J}$ was established in Remark 4.18. Now, let $v \in \mathcal{R} \otimes \mathbb{Q}$ of degree r; we argue as in Remark 4.18. $\mathcal{R}^i$ is constant for $i \neq n-1$, and likewise $H^i(\mathcal{Y})$ is an invariant G-module, where $G = Gal(k(t)^{sep}|k(t))$, and $H^{n-1}(\mathcal{Y}) = \iota_\eta^* H^{n-1}(X) \oplus V(\mathcal{Y})$, the first summand being $H^{n-1}(\mathcal{Y})^G$ by Corollary 4.10. One should add that, since $V(\mathcal{Y})$ is a self-dual G-module (via cup-product), $V(\mathcal{Y})$ cannot have a G-invariant quotient. Now, given the interpretations we have for $H^0(\mathcal{R}^i)$ and $H^2(\mathcal{R}^i)$ under (A) (see e.g. [6, XVIII. Lemme 5.6.9]), namely $H^i(\mathcal{Y})^G$ and $H^i(\mathcal{Y})_G$, one has: $[v_\eta] H^i(\mathcal{Y}) \subset H^{i+2r}(\mathcal{Y})^G$ for $i \neq n-1$ (as $[v_\eta] = [v]_{\mathcal{Y}}$ induces G-equivariant linear maps on cohomology). One also obtains, for $r \neq 0$, $[v]_{\mathcal{Y}} V(\mathcal{Y}) = 0$ by the above. Thus, if $\mathcal{A}_r$ denotes the correspondences in $\mathcal{A}$ of degree r, we have seen $\mathcal{I}_r = \mathcal{J}_r$ for $r \neq 0$; the inclusion $\mathcal{I}_0 \subset \mathcal{J}_0$ is also clear (see for instance Remark 4.18); $\mathcal{J} \subset \mathcal{J}'$ follows from the preceding discussion. By the same token, $\mathcal{I}_r = \mathcal{J}_r$ for $r \neq 0$. Now, the subspace $\mathcal{J}'_0 \subset \mathcal{R} \otimes \mathbb{Q}$ given by $\{v \in \mathcal{R} \otimes \mathbb{Q}\colon [v]_{\mathcal{Y}} H^*(\mathcal{Y}) \subset V(\mathcal{Y})\}$ is the preimage of an ideal of $\operatorname{End} h(\mathcal{Y})$ (as follows from the above arguments), and $\mathcal{J}'$ is its homomorphic image in $\mathcal{A}$, which completes (i).

If $H^1(\mathcal{R}^{n-1}) = 0$, then $\mathcal{J}' \subset \mathcal{J}$, which settles (ii).

To prove (iii), recall that if n is even or $\operatorname{char} k \neq 2$, then all the singularities of ρ are non-degenerate quadratic singularities of fibres, and the monodromy representation of $\pi_1^{alg}(B, \bar{\eta})$ (B being the smooth locus of ρ) on $V(\mathcal{Y})$ is absolutely irreducible [6, esp. XVIII Cor. 6.7]. As a result, the $\pi_1(B)$-submodule $[w_0]_{\mathcal{Y}} V(\mathcal{Y})$ is either 0 or $V(\mathcal{Y})$, and so there are two possibilities for the inclusion of $\mathbb{Q}_\ell$-sheaves $w_0 \mathcal{E}^{n-1} \hookrightarrow \mathcal{E}^{n-1}$: either the image or the cokernel of this inclusion are skyscraper sheaves. Since $H^1(\mathcal{R}^{n-1}) = H^1(\mathcal{E}^{n-1}) \neq 0$, we have

$$w H^1(\mathcal{R}^{n-1}) = 0 \quad \Leftrightarrow \quad [w_0]_{\mathcal{Y}} V(\mathcal{Y}) = 0,$$

the other case being $w H^1(\mathcal{R}^{n-1}) = H^1(\mathcal{R}^{n-1})$. The first assertion of (iii) follows from this argument, as $[v]_{\mathcal{Y}} = 0$ if and only if $[j_* v]$ acts as 0 on $Gr_F H^*(\tilde{X})$.

To prove the second assertion of (iii), assume $B(X)$. Then the projector $e_{V(\mathcal{Y})}$ is algebraic by Proposition 7.10, and one may choose a lifting u of this algebraic class. u acts as 0 on $H^\epsilon(\mathcal{R}^i)$

unless $\epsilon = 1$ and $i = n - 1$, in which case it acts as the identity, and so we have $u \in \mathcal{J}'$ but not in $\mathcal{J}$.

Let us prove (iv); by Proposition 4.17, $H^1(\mathcal{R}^{n-1}) \simeq P^n(X) \oplus V(\Delta)$. If $P^n(X) \neq 0$ there is nothing to prove; if $P^n(X) = 0$ the assertion follows from the next elementary lemma.

Lemma 4.21. *(Compare [6, XVIII Lemme 6.4.2].) With the notations and hypotheses of Proposition* 4.20(iv) *(assuming $n \geqslant 3$), let $d \in \mathbb{N}$. Let $Y(d)$, $Y'(d)$ denote degree-d hypersurface sections intersecting transversally, and let $\Delta(d) = Y(d) \cap Y'(d)$. Then $b_{n-2}(\Delta(d))$ is a polynomial of degree n in d.*

Proof. Let $c(X)$, $c(\Delta)$ be the total Chern classes of X, Δ and let $j\colon \Delta \subset X$ denote the canonical inclusion. Let $\int_X$ denote the trace map on X, and $H = c_1(\mathcal{O}_X(1))$ with the present polarisation. Then, using $j_* j^* \alpha = d^2 H^2 \bullet \alpha$, we obtain:

$$\chi\bigl(\Delta(d)\bigr) = \int\limits_{\Delta(d)} c\bigl(\Delta(d)\bigr) = \int\limits_{\Delta(d)} j^* \frac{c(X)}{(1 + d\cdot H)^2} = \int\limits_X \frac{d^2 H^2 c(X)}{(1 + d\cdot H)^2},$$

which is a polynomial in d of degree precisely n. Isolating yields $b_{n-2}(\Delta(d)) = (-1)^{n-2}\chi(\Delta(d)) + 2\sum_{i \geqslant 1}(-1)^i b_{n-2-i}(X)$, thus establishing the lemma. □

Taking $d \gg 0$, the d-uple embedding of $X \subset \mathbb{P}$ satisfies the hypotheses of (iii) (and **(A)**, see [6, XVIII Lemme 6.4.2]), which settles (iv).

It remains to prove (v). Assume n even or $\operatorname{char} k \neq 2$. This means that $V(\mathcal{Y})$ is absolutely irreducible as a G-module, hence $v_\eta V(\mathcal{Y}) = 0$ or $V(\mathcal{Y})$. Arguing as in (i), (v) is completed. Proposition 4.20 is now established. □

Corollary 4.22. *Under the assumptions of Proposition* 4.20*; if n is even or* $\operatorname{char} k \neq 2$*, and $H^1(\mathcal{R}^{n-1}) \neq 0$, then the restriction map*

$$\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{Q} \twoheadrightarrow A^{n-1+*}(\mathcal{Y} \times \mathcal{Y}) = \operatorname{End} h(\mathcal{Y})$$

defined in Proposition 4.14 *has kernel $\mathcal{I}^0$, which is precisely the preimage of $\mathcal{I} \subset \mathcal{A}$ via the morphism*

$$\lambda : \mathcal{R} \otimes \mathbb{Q} \twoheadrightarrow \mathcal{A} \subset \operatorname{End} h(\tilde{X}),$$

and therefore factors through a ring homomorphism

$$res_{\mathcal{Y}} : \mathcal{A} \twoheadrightarrow A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$$

defined by $\lambda(v) = [j_* v]$, whose kernel is $res_{\mathcal{Y}} = \mathcal{I}$.

Proof. The corollary follows from Proposition 4.20(v): write $\mathcal{A} = \mathcal{R} \otimes \mathbb{Q}/\mathfrak{b}$. We observe that $\mathfrak{b} \subset \mathcal{I}^0$, as the arguments in the proof of Proposition 4.20 imply that this must be the case. One has $\mathfrak{b} \subset \lambda^{-1}(\mathcal{J}) = \lambda^{-1}\mathcal{I} = \mathcal{I}^0$, by Proposition 4.20(iii), so the quotient map $\mathcal{R} \otimes \mathbb{Q} \twoheadrightarrow \operatorname{End} h(\mathcal{Y})$ factors through $\mathcal{A}$, inducing a ring epimorphism $res_{\mathcal{Y}}$ whose kernel is precisely $\mathcal{I}^0/\mathfrak{b} = \mathcal{I}$. The proof is thus complete. □

Remark 4.23. By Proposition 4.20 above, there is a ring epimorphism

$$\varphi = \varphi_{\mathcal{Y}} : A^{n-1+*}(\mathcal{Y} \times \mathcal{Y}) \twoheadrightarrow \mathcal{A}/\mathcal{I},$$

which is an isomorphism if n is even or $\operatorname{char} k \neq 2$ by Corollary 4.22. In general the source of $\varphi_{\mathcal{Y}}$ equals $\mathcal{R} \otimes \mathbb{Q}/\mathcal{I}^0$, and $\mathcal{A}/\mathcal{I} = \mathcal{R}_\rho \otimes \mathbb{Q}/(\mathcal{I}^0 + \mathfrak{b})$. It is not hard to see from the proof of Proposition 4.20 that $\mathfrak{b}_r = \mathcal{I}_r^0 = \lambda^{-1}(\mathcal{J}_r) = \lambda^{-1}(\mathcal{J}_r')$ for $r \neq 0$, with the notations of Corollary 4.22.

The next corollary circumvents the possible non-isomorphy of φ for the purposes of this paper.

Corollary 4.24. *Let $\mathfrak{a} \subset A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$ be the ideal of self-correspondences w of $\mathcal{Y}$ such that $wH^*(\mathcal{Y}) \subset V(\mathcal{Y})$. One has a ring isomorphism induced by φ above:*

$$A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})/\mathfrak{a} \xrightarrow{\sim} \mathcal{A}/\mathcal{J}'.$$

Consider a graded unital subalgebra $\mathcal{B} \subset A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$ such that $\mathcal{B} \cap \mathfrak{a} = 0$. Then φ yields an isomorphism $\mathcal{B} \simeq \varphi(\mathcal{B}) \subset \mathcal{A}/\mathcal{I}$, which maps isomorphically after composing with the quotient map $\mathcal{A}/\mathcal{I} \to \mathcal{A}/\mathcal{J}'$.

Proof. As we have seen in the proof of Proposition 4.20, in general (and under **(A)**) one has $\mathfrak{b} \subset \bar{r}_\eta^{-1}(\mathfrak{a})$ where $\bar{r}_\eta$ was defined in Proposition 4.14. The corollary follows. □

5. The relative projectors

In this section we assume **(A)**. We have seen in Lemma 2.1 that, if $C(X)$ holds, then the ring of correspondences of X, $A^{\dim X+\bullet}(X \times X)$ decomposes through the adjoint action of H_X, $u \mapsto [H, u]$; the degree-0 correspondences are exactly those commuting with H_X, or equivalently, with the Künneth projectors π_X^i for all i. Assuming $C(\mathcal{Y})$, we wish to translate this situation into the relative context presented in Section 4, creating relative analogues π_ρ^i, H_ρ of π^i and of H, supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$, which will not be strictly possible unless **(A)** is assumed, otherwise one has to assume $B(\mathcal{Y})$. We now deal with the first case. The relative projectors we are to construct yield a splitting of the Leray filtration, and under the hypotheses of Proposition 4.20(iii) a section of the ring epimorphism $res_{\mathcal{Y}}$ defined in Corollary 4.22.

Lemma 5.1. *Assume $C(\mathcal{Y})$. Let $\pi'^i \in A^n(\tilde{X} \times \tilde{X})$ be liftings of $\pi_{\mathcal{Y}}^i$. Then π'^i are such that*

$$\pi'^i | Gr_F^\epsilon H^j(\tilde{X}) = \delta_{i,j-\epsilon}$$

for all $0 \leqslant i, j \leqslant 2n-2$ and $\epsilon = 0, 1, 2$. The restriction of π'^j to $F^2H^*(\tilde{X})$ is a projector which yields 0 on $F^2H^j(\tilde{X})$ if $j \neq i+2$ and the identity if $j = i+2$. The restriction of π'^j to F_ρ^2 is therefore independent of the lifting chosen.

Proof. The proof is laid out in 4.2—see Remark 4.18. If $\pi'^i \in \mathcal{A}$ is a lifting of $\pi_{\mathcal{Y}}^i$, then $\pi'^i | H^\epsilon(R^k\rho_*\mathbb{Q}_\ell) = \delta_{i,k}$ for all i, k, and the lemma follows. □

Consider a set of liftings π'^i of $\pi_{\mathcal{Y}}^i$ such that ${}^t\pi'^i = \pi'^{2n-2-i}$. With this choice, the following proposition is the relative equivalent to Lemma 2.8.

Proposition 5.2. *Define $H' := \sum(n-1-i)\pi'^i$ with π'^i as above. Then H' is skewsymmetric, and its semisimplification H_ρ, which is also skew-symmetric, satisfies $H_\rho = \sum(n-1-i)\pi_\rho^i$ where π_ρ^i are liftings of $\pi_{\mathcal{Y}}^i$ and form a complete orthogonal system of projectors, and ${}^t\pi_\rho^i = \pi_\rho^{2n-2-i}$. The projectors $\pi^{i,e} = \pi_{\tilde{X}}^{i+e}\pi_\rho^i$ split the Leray spectral sequence of ρ, and the following decompositions hold:*

$$\pi_{\tilde{X}}^i = \pi^{i,0} + \pi^{i-2,2} \quad \text{for } i \neq n, \qquad \pi_{\tilde{X}}^n = \pi^{n,0} + \pi^{n-1,1} + \pi^{n-2,2} \quad \text{and} \quad \pi_\rho^i = \pi^{i,0} + \pi^{i,2}$$

for $i \neq n-1$. If $i = n-1$ then $\pi_\rho^{n-1} = \pi^{n-1,0} + \pi^{n-1,1} + \pi^{n-1,2}$. The splitting depends on the choice of π'^i.

Proof. Define $H' := \sum(n-1-i)\pi'^i$; by construction ${}^tH' = -H'$ and H' is a lifting of $H_{\mathcal{Y}}$. By Lemma 5.1, H' acts on $Gr_F H^*(\tilde{X})$ as $(n-1-i)id$ on $H^e(\mathcal{R}^i)$, so a power of the polynomial $p(x) = x\prod_{i=1}^{n-1}(x^2 - i^2)$ annihilates H'; more precisely, $p(H') \in \mathcal{J}$, hence $p(H')^3 = 0$. The semisimple component H_ρ of the Jordan decomposition of H' is necessarily skew-symmetric, and its minimal polynomial is precisely $p(x)$. Thus the projectors π_ρ^i associated with the eigenspaces corresponding to $n-1-i$ are polynomials in H_ρ, hence in H', and by Section 4.1 they are liftings of $\pi_{\mathcal{Y}}^i$. To show ${}^t\pi_\rho^i = \pi_\rho^{2n-2-i}$ simply transpose the identity $H_\rho = \sum(n-1-i)\pi_\rho^i$; equating yields $-(n-1-i)\pi_\rho^i = (i-n+1){}^t\pi_\rho^{2n-2-i}$, which in turn implies ${}^t\pi_\rho^i = \pi_\rho^{2n-2-i}$ for $i \neq n-1$; the remaining projector π_ρ^{n-1} must be therefore symmetric.

The set of projectors obtained clearly provides a splitting of the Leray filtration $F_\rho^* H^*(\tilde{X})$, and the rest follows. □

Observation–Definition. Let $\tilde{u}$ be a correspondence of degree r of $\mathcal{Y}$. Then

$$\tilde{u} = \sum \pi_{\mathcal{Y}}^{i+2r}\tilde{u}\pi_{\mathcal{Y}}^i. \tag{16}$$

This goes along with (and in fact implies) the commutation relation in Lemma 2.1. We define for each $u \in \mathcal{A}$ the following element of $\mathcal{A}$:

$$u_\rho := \sum \pi_\rho^{i+2r} u \pi_\rho^i. \tag{17}$$

It is clear by construction that $u_\rho - u \in \mathcal{J}$. If u is a correspondence of degree r on $\mathcal{Y}$, we will define u_ρ to be u'_ρ for u' a lifting of u in $\mathcal{A}$. Later we will see that this definition is consistent.

Lemma 5.3. *The map $\psi : \mathcal{A} \to \mathcal{A}$ defined by $u \mapsto u_\rho$ satisfies $\mathcal{J} = \operatorname{Ker}\psi$; in other words, $u_\rho = v_\rho$ if and only if $u - v$ induces 0 on $Gr_{F_\rho}^\bullet$. The image $\psi(\mathcal{A})$ in degree r consists of the $w \in \mathcal{A}$ such that $\pi_\rho^{i+2r} w = w\pi_\rho^i$ for all i. The map ψ is a linear projector which induces a section σ of the natural quotient map $\mathcal{A} \to \mathcal{A}/\mathcal{J}$* à la *Wedderburn–Malcev, and commutes with transposition. As a result we have a well-defined ring homomorphism*

$$A^{n-1+*}(\mathcal{Y} \times \mathcal{Y}) \to \psi(\mathcal{A})$$

defined by $u \mapsto u_\rho$, which agrees with the homomorphism $\psi \circ proj_{\mathcal{J}} \circ \varphi$.

Proof. It is clear that $\mathcal{J} = \operatorname{Ker}\psi$. The image of ψ is easily characterised as the subspace of u such that $\psi(u) = u$ (easily seen to agree with the description $u\pi_\rho^i = \pi_\rho^{i+2r}u$ for u of degree r), whence $\psi^2 = \psi$. By Proposition 5.2, $\psi({}^t u) = {}^t\psi(u)$. Finally, the terms $v_\rho \circ u_\rho$ and $(v \circ u)_\rho$ differ by an element of $\mathcal{J} \cap \operatorname{Im}\sigma_{\mathcal{Y}} = (0)$, which shows that ψ is a ring homomorphism. ψ clearly induces a section $\mathcal{A}/\mathcal{J} \to \mathcal{A}$ of the quotient map, which gives rise to the map $u \mapsto u_\rho$ with source $A^{n-1+*}(\mathcal{Y} \times \mathcal{Y})$. $\square$

5.1. *Relative projectors without* **(A)**

We now will see how far we can go assuming $C(\mathcal{Y})$ for an arbitrary Lefschetz fibration ρ on X; we can obtain a set of relative projectors, but their action on $H^n(\tilde{X})$ is unclear.

As above, we consider a lifting H' of $H_{\mathcal{Y}}$ that is skew-symmetric, and consider its semisimplification; without loss of generality we may assume that H' mas minimal polynomial $p(x) = x\prod_{k=1}^{n-1}(x^2 - k^2)$ by substituting $q(H')$ for H' for a suitable (odd) polynomial $q(H')$. We know that the action of H' on $\tilde{F}^0/\tilde{F}^1$ and on $\tilde{F}^2 = F^2$ is independent of the lifting chosen, but then the action on $\tilde{F}^1H^n(\tilde{X})/\tilde{F}^2H^n(\tilde{X})$ is unclear without further assumptions on $\mathcal{Y}$. Denote $W := \bigoplus_{i\neq n} H^i(\tilde{X})$; W is a self-dual subspace of $H^*(\tilde{X})$.

For $i \leqslant n-1$, we define $\tilde{\pi}^i$ to be the projector corresponding to the eigenspace of the eigenvalue $n-1-i$ of $\tilde{H}$, which will be a correspondence supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$; denote $\pi^{i,\epsilon} := \tilde{\pi}^i\pi_{\tilde{X}}^{i+\epsilon}$. One has $\tilde{\pi}^i|W = (n-1-1)\pi^{i,0} + (n+1-i)\pi^{i,2}$ for $i \neq n$ and also $F^2H^n(\tilde{X}) \subset \tilde{\pi}^{n-2}H^n(\tilde{X})$; here equality is not granted without further assumptions on $\mathcal{Y}$, nor can it be decided whether $\tilde{\pi}^jH^n(\tilde{X})$ is zero or not for $|j-n+1| \geqslant 2$.

Definition. Let $u \in \mathcal{A}$ be a relative correspondence of degree $r \neq 0$. We define u_ρ to be the image of $\sum \tilde{\pi}^{i+2r}u\tilde{\pi}^i$ in $\operatorname{End} Gr_{\tilde{F}^*}H^*(\tilde{X})$.

We now prove Remark 4.19.

Proposition 5.4. *Consider a correspondence v supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$ and denote $u = [j_*v]$. Identify $\tilde{F}^0/\tilde{F}^1 = H^*(\mathcal{Y})^G$ and $F^2 = H^{*-2}(\mathcal{Y})_G$. Then the action of u on $\tilde{F}^0/\tilde{F}^1$ agrees with the action of $[v]_{\mathcal{Y}}$ on $H^*(\mathcal{Y})^G$, and this is also the case for $F^2 = H^{*-2}(\mathcal{Y})_G$.*

Choose an open subset ν: $B \subset \mathbb{P}^1$ within the smooth locus of ρ. If **(A)** *holds for ρ, then u acts on $F^1/F^2H^n(\tilde{X}) = H^1(\mathcal{R}^{n-1}) = H^1(\nu_*\nu^*\mathcal{R}^{n-1})$ as the map induced by v on $\mathcal{R}^{n-1}$ above $\mathcal{U}$, and in fact depends only on $[v]_{\mathcal{Y}}$.*

Proof. First of all, consider a generically smooth map $f : V \to C$, with V, C projective smooth, C a curve and V of dimension N. Denote $\mathcal{R}_f^i := R^if_*(\mathbb{Q}_\ell)$. Let $\nu : \mathcal{U} \hookrightarrow C$ be the natural inclusion, where $\mathcal{U}$ is within the smooth locus of f. One has a bigraded algebra $E'^{e,i}(f) := H^e(\nu_*\nu^*\mathcal{R}_f^i)$, and a natural map of bigraded algebras $E_2^{e,i}(f) \to E'^{e,i}(f)$. The bigraded algebra $E'(f)$ satisfies Poincaré-like Duality, i.e. the cup-product

$$E'^{e,i}(f) \otimes E'^{2-e,2N-2-i}(f) \to E'^{2,2N-2}(f) \cong \mathbb{Q}_\ell$$

is a perfect pairing. If f has connected fibres this is just the usual Poincaré Duality in [11]. Otherwise, one needs to take the Stein factorization of f, $U \xrightarrow{f'} C' \xrightarrow{\sigma} C$ (where C' is smooth) and compare $E_2(f)$, $E_2(f')$ and $E'(f)$, $E'(f')$ choosing $\mathcal{U}' := \sigma^{-1}(\mathcal{U}) \subset C'$ in the definition of $E'(f)$. If $\mathcal{V}$ is the generic fibre of f, then $E'^{2,2N-2}(f) = H^{2N-2}(\mathcal{V})_G = [H^0(\mathcal{V})^G]^\vee = \mathbb{Q}_\ell$ where $G = \pi_1(C)$, by irreducibility of V, hence of $\mathcal{V}$ over $k(C)$.

We now wish to prove that, for $\tilde{F}^0 H^*(\tilde{X})/\tilde{F}^1 H^*(\tilde{X})$ and $F^2 H^*(\tilde{X})$, the action of u agrees with the action of $[v]_{\mathcal{Y}}$ on the generic fibre $\mathcal{Y}$, under the above identifications. Fix a De Jong alteration [3] $\beta : Z \to \tilde{X} \times_{\mathbb{P}^1} \tilde{X}$, and define $\lambda : Z \to \tilde{X} \times \tilde{X}$ to be $\lambda = j \circ \beta$. The map $CH_*(Z) \otimes \mathbb{Q} \to CH_*(\tilde{X} \times_{\mathbb{P}^1} \tilde{X}) \otimes \mathbb{Q}$ is surjective, so we can write $u = [j_* v] = \lambda_*(v')$ for $v' \in A^*(Z)$.

Denote $p_i : \tilde{X} \times_{\mathbb{P}^1} \tilde{X} \to \tilde{X}$ $(i = 1, 2)$ and $q_i = p_i \circ \beta$. Let φ denote the structure map of $Z/\mathbb{P}^1$. One may write $ux = \lambda_*(v') = q_{2*}(v' q_1^*(x))$, so for $x \in \tilde{F}^0/\tilde{F}^1 H^i(\tilde{X})$ and $y \in F^2 H^{2n-2r-i}(\tilde{X})$ one has

$$\langle u(x), y \rangle_{\tilde{X}} = \langle v' \bullet q_1^* x \bullet q_2^* y \rangle_Z; \tag{18}$$

we henceforth use the same notations for x, y or their pullbacks and their respective images in $E'(\bullet)$, $H^*(\mathcal{Y})^G$, etc. via the identifications given. One has $q_1^* x \bullet q_2^* y \in F_\varphi^2 H^*(Z)$, so the degree in (18) depends on v' (mod $F^1 H^*(Z)$). Let $w \in H^0(R_\varphi^*)$ be a preimage of the class of v' in $F_\varphi^0 / F_\varphi^1$; w maps to $H^0(\nu_* \nu^* R^*) = H^*(\mathcal{Z})^G$ as the restriction $[v'_\eta]$ of $[v']$. Now, since $E_2^{2,2(2n-1)-2}(\varphi) = E'^{2,2(2n-1)-2}(\varphi) = \mathbb{Q}_\ell$, one has

$$\langle v' \bullet q_1^* x \bullet q_2^* y \rangle_Z = \langle w \bullet q_1^* x \bullet q_2^* y \rangle_{E_2(\varphi)} = \langle [v'_\eta] \bullet q_1^* x \bullet q_2^* y \rangle_{E'(\varphi)},$$

and this in turn equals

$$\langle [v'_\eta] \bullet q_{1\eta}^* x \bullet q_{2\eta}^* y \rangle_{\mathcal{Z}}.$$

This computation on $H^*(\mathcal{Z})$ can be pushed forward to $H^*(\mathcal{Y} \times \mathcal{Y})$ via the map $\beta_\eta : \mathcal{Z} \to \mathcal{Y} \times \mathcal{Y}$

$$\langle [v'_\eta] \bullet q_{1\eta}^* x \bullet q_{2\eta}^* y \rangle_{\mathcal{Z}} = \beta_{\eta *} \langle [v_\eta] \bullet p_{1\eta}^* x \bullet p_{2\eta}^* y \rangle_{\mathcal{Y} \times \mathcal{Y}},$$

which in turn equals $\langle [v_\eta] x, y \rangle_{\mathcal{Y}}$ as desired. The proof for F^2 is identical. Assume **(A)**; if the degree of u is not 0, the statement is obvious. We are going to prove that the action of $u = \lambda_* v'$ of degree 0 is induced on $F^1/F^2 H^n(\tilde{X}) \cong H^1(\mathcal{R}^{n-1}) = H^1(\nu_* \nu^* \mathcal{R}^{n-1})$ by the class v on the fibres of ρ over an open subset ν: $B \subset \mathbb{P}^1$ where φ (and hence also ρ) is smooth; we are going to use the Poincaré-type duality results available in the present setting. The proof is similar as above. Note that the pairing

$$Gr_F^1\big(H^n(\tilde{X})\big) \otimes Gr_F^1\big(H^n(\tilde{X})\big) \to F^2 H^{2n}(\tilde{X}) = H^{2n}(\tilde{X})$$

is perfect and agrees with $E'^{1,n-1}(\rho) \times E'^{1,n-1}(\rho) \to E'^{2,2n-2}(\rho)$. As above, making the corresponding identifications, we have

$$\langle ux, y \rangle_{\tilde{X}} = \langle ux, y \rangle_{E'(\rho)} = \langle v' \bullet q_1^* x \bullet q_2^* y \rangle_{E'(\varphi)} = \langle [v] \bullet p_1^* x \bullet p_2^* y \rangle_{E'(\rho_2)}$$

where the image of v' in $E'^{0,2n-2}(\varphi)$ is also denoted by v' and $[v]$ denotes the image of the Chow class v within $H^0(\nu_*\nu^* R^{2n}_{\rho_2})$. This in turn equals

$$\langle [v]_{\mathcal{Y}} \bullet \bigl(p_1^*(x) \bullet p_2^*(y)\bigr)\rangle_{\mathcal{Y}\times\mathcal{Y}},$$

where $p_1^*(x) \bullet p_2^*(y)$ is seen as a class in $H^{2n-2}(\mathcal{Y} \times \mathcal{Y})_G$. The proposition is thus established. □

Remark 5.5. Assume that u, u' are two liftings of the same class $v \in \operatorname{End} h(\mathcal{Y})$ of degree $r \neq 0$. Then $u'_\rho - u_\rho$ belongs to the nilpotent ideal $\tilde{\mathcal{J}} \subset \mathcal{A}$ defined by $\tilde{\mathcal{J}} := \{u \in \mathcal{A}\colon u\tilde{F}^i \subset \tilde{F}^{i+1}\}$. If $B(\mathcal{Y})$ and **(A)** are not assumed, however, $u'_\rho - u_\rho$ need not be zero, and not even nilpotent if $r = 0$. Indeed, without these assumptions the projectors $\pi^i_\rho \pi^{i+\epsilon}_{\tilde{X}}$ need not split the filtration $\tilde{F}^* H^*(\tilde{X})$, as the inclusion $\tilde{F}^2 H^n(\tilde{X}) \subset \pi^{n-2}_\rho H^n(\tilde{X})$ could be strict.

6. A relative $\mathfrak{sl}_2$-triple

In the beginning of this section we assume **(A)**. We have obtained a set of relative Künneth projectors under the hypothesis $C(\mathcal{Y})$. In this section we assume $B(\mathcal{Y})$ and we construct relative operators ${}^c\Lambda_\rho$, Λ_ρ lifting ${}^c\Lambda_{\mathcal{Y}}$, $\Lambda_{\mathcal{Y}}$, first under **(A)** and then in the general case. These will give rise to a relative $\mathfrak{sl}_2$-triple ${}^c\Lambda_\rho, L_\rho, H_\rho$ supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$, whose action on $H^*(\tilde{X})$ will be exploited later.

Proposition 6.1. *The following assertions hold.*

(1) *For any lifting ${}^c\Lambda'$ of ${}^c\Lambda_{\mathcal{Y}}$, the correspondence ${}^c\Lambda_\rho = \sum \pi^{i-2c}_\rho \Lambda' \pi^i_\rho$ is symmetric and independent of the lifting ${}^c\Lambda'$ chosen.*
(2) *The operator ${}^c\Lambda_\rho$ satisfies ${}^c\Lambda_\rho \pi^{i,2} \subset \operatorname{Im} \pi^{i-2,2}$ and ${}^c\Lambda_\rho \pi^{i,0} \subset \operatorname{Im} \pi^{i-2,0}$. In fact ${}^c\Lambda_\rho \pi^{i,0} = \pi^{i-2,0\,c}\Lambda_\rho$ and ${}^c\Lambda_\rho \pi^{i,2} = \pi^{i-2,2\,c}\Lambda_\rho$.*

Proof. (1) follows from Lemma 5.3. (2) follows directly from Proposition 4.15 and Lemma 5.3. Alternatively, one may consider ${}^c\Lambda_{\mathcal{Y}}$, L and the algebra generated by these two classes, and then apply Corollary 4.24 and Lemma 5.3. □

Finally we obtain the desired $\mathfrak{sl}_2$-triple.

Proposition 6.2. *We have a relative $\mathfrak{sl}_2$-triple ${}^c\Lambda_\rho, L_\rho, H_\rho$. A relative Lefschetz isomorphism holds:*

$$L^i_\rho : \operatorname{Im} \pi^{n-1-i}_\rho \to \operatorname{Im} \pi^{n-1+i}_\rho$$

for $1 \leqslant i \leqslant n-1$. The projectors p^i_ρ are algebraic for $i \leqslant n-1$, and we have symmetric operators p^{n-1+j}_ρ derived from $p^{n-1+j}_{\mathcal{Y}}$ for $0 \leqslant j \leqslant n-1$. The map $u \mapsto u_\rho$ yields an isomorphism of rings $\mathbb{Q}\langle L_\rho, \Lambda_\rho\rangle \cong \mathbb{Q}\langle L_{\mathcal{Y}}, \Lambda_{\mathcal{Y}}\rangle$ which preserves transposition.

Proof. The $\mathfrak{sl}_2$-identities $[H_\rho, {}^c\Lambda_\rho] = 2\,{}^c\Lambda_\rho$, $[H_\rho, L_\rho] = -2L_\rho$ and $[{}^c\Lambda_\rho, L_\rho] = H_\rho$ and the isomorphism between $\mathbb{Q}\langle L_\rho, \Lambda_\rho\rangle$ and $\mathbb{Q}\langle L_{\mathcal{Y}}, \Lambda_{\mathcal{Y}}\rangle$ follow from Lemma 5.3 and Corollary 4.24. The operators $\frac{1}{m^i} L^i_{\tilde{X}}$, L^i and L^i_ρ induce the same map on $Gr^{\bullet}_{F_\rho}$ by Proposition 4.3 and Lemma 4.6. The 'relative Lefschetz isomorphism' can be checked by passing to Gr_{F_ρ}, or simply by using the identities $(\Lambda^i_\rho L^i_\rho - 1)\pi^{n-1-i}_\rho = 0$ for $0 \leqslant i \leqslant n-2$. □

Now we can view the Lefschetz theory of $\mathcal{Y}$ within $H^*(\tilde{X})$.

6.1. In absence of **(A)**

Assume $B(\mathcal{Y})$, and ρ arbitrary. We first construct a decent set of relative projectors, as done in Section 5 under **(A)**. Consider the identity

$$\left[{}^c\Lambda_{\mathcal{Y}}, L\right] = H_{\mathcal{Y}}.$$

Now we construct H_ρ: take a lifting ${}^c\Lambda'$ of ${}^c\Lambda_{\mathcal{Y}}$ which is symmetric, and define $H' := [{}^c\Lambda', L]$. The following proposition holds:

Proposition 6.3. *Let ρ be an arbitrary Lefschetz fibration. Assume $B(\mathcal{Y})$ and consider H' as above. H' is skew-symmetric, and so is its semisimplification, which we denote by H_ρ. H' satisfies $H' \operatorname{Ker} k^* = H' \operatorname{Im} k_* \subset \operatorname{Im} k_*$, and so does H_ρ. Furthermore H_ρ decomposes as*

$$H_\rho = \sum (n-1-i)\pi^i_\rho,$$

where π^i_ρ are projectors supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$. The projectors π^i_ρ act trivially on $H^j(\tilde{X})$ for $j \neq i, i+1, i+2$, and $\pi^{i-2}_\rho H^i(\tilde{X}) = k_* H^{i-2}(Y)$ for $2 \leqslant i \leqslant 2n$. The projectors $\pi^i_\rho \pi^{i+e}_{\tilde{X}}$ split the filtration $\tilde{F}^* H^*(\tilde{X})$.

Proof. The only thing we need to remark is that by construction H' acts like 0 on $\tilde{F}^1 H^n(\tilde{X}) / \tilde{F}^2 H^n(\tilde{X})$, and since H_ρ restricts to $1_{F^2 H^n(\tilde{X})}$ on $\tilde{F}^2 H^n(\tilde{X})$ and to -1 on $\tilde{F}^0 H^n(\tilde{X}) / \tilde{F}^1 H^n(\tilde{X})$, one has $\pi^{n-2}_\rho H^n(\tilde{X}) = F^2 H^n(\tilde{X})$ by a dimension count. The proposition follows. □

Once we fix a choice of H_ρ as above, we denote $\upsilon_\rho := \upsilon_{H_\rho}$. The Lefschetz theory of $\mathcal{Y}$ thus translates to this relative setting, and we get

Proposition 6.4. *Let ρ be an arbitrary Lefschetz fibration. We have operators $\Lambda_\rho, L_\rho, {}^c\Lambda_\rho$ and primitive operators p^i_ρ satisfying the same properties as in Proposition* 6.2.

Proof. Remember that the operators $p^i_{\mathcal{Y}}$ are non-commutative polynomials in $\Lambda_{\mathcal{Y}}$ and $L_{\mathcal{Y}}$ (see Proposition 2.5). All properties hold over $\bigoplus_{e=0}^2 \tilde{F}^e / \tilde{F}^{e+1}$; now, the projectors $\pi^{i+e}_{\tilde{X}} \pi^i_\rho$ do split the filtration $\tilde{F}^* H^*(\tilde{X})$ by Proposition 6.3 above, and gone is the obstruction encountered in Remark 5.5, so any self-correspondence υ of $\mathcal{Y}$ of degree $\neq 0$ yields a unique υ_ρ. Thus we have unique operators Λ_ρ and L_ρ (and ${}^c\Lambda_\rho$) corresponding to our choice of H_ρ, which establishes the proposition. □

7. The Main Theorem

This section is devoted to proving the following.

Main Theorem. *Let X be a smooth, projective variety of dimension n. Assume the Lefschetz standard conjecture for the generic fibre $\mathcal{Y}/k(t)$ of a Lefschetz pencil. Then $\Lambda - p^{n+1}$ is algebraic.*

We will prove it in a series of steps, obtaining the algebraicity of the Künneth projectors $\pi^i_{\tilde{X}}$ for $i \neq n-1, n, n+1$ in the course of our proof.

7.1. The algebraicity of some projectors

For simplicity, we still suppose k algebraically closed. We start by proving the following.

Proposition 7.1. *Assume $B(\mathcal{Y})$, and ρ arbitrary. Then the Künneth projectors π^i_X for $i \leqslant n-2$ are algebraic (hence also π^i_X for $i \geqslant n+2$) and so are the primitive projectors $p^0, \dots, p^{n-2}$.*

A couple of lemmas will be required to establish the proposition.

Lemma 7.2. *The following statements hold (ρ arbitrary).*

(i) *The identity $\iota_* H^{i-2}(Y) = LH^{i-2}(X)$ holds for all $0 \leqslant i \leqslant 2n$, and $\iota_*: H^{i-2}(Y) \to H^i(X)$ is injective for $i \leqslant n$. For all $i \leqslant n$,*

$$\operatorname{Im}\bigl(\pi^i_X - p^i_X\bigr) = LH^{i-2}(X).$$

(ii) *For all $i > n$, $\operatorname{Im}(\pi^i_X - {}^t p^{2n-i}_X) = L^{i-n+1} H^{2n-i-2}(X)$.*

(iii) *Suppose $B(Y)$ holds for Y a smooth hyperplane section of X. Then for $0 \leqslant i \leqslant n$,*

$$\pi^i_X - p^i_X = \iota_* \Lambda_Y \pi^i_Y \iota^*$$

is algebraic. Thus the transposed operators ${}^t(\pi^i_X - p^i_X) = \pi^{2n-i}_X - L^{n-i} p^{2n-i}_X$ are algebraic for $0 \leqslant i \leqslant n$.

(iv) *The hypothesis $B(\mathcal{Y})$ of the Main Theorem implies $B(Y)$ for any smooth member Y of ρ.*

Proof. Statements (i)–(iii) are straightforward. To prove (iv), one need remark that, for the inclusion $\nu: \mathcal{U}_0 \subset \mathbb{P}^1$ where $\mathcal{U}_0$ is the smooth locus of ρ, the sheaves $\nu_* \nu^* \mathcal{R}^i$ are constant for $i \neq n-1$ and $\mathcal{R}^{n-1} = \mathcal{E}^{n-1} \oplus P^{n-1}(X)_{\mathbb{P}^1} \oplus L\mathcal{R}^{n-3}$ (where $\mathcal{E}^{n-1} \oplus P^{n-1}(X)_{\mathbb{P}^1} = \mathcal{P}^{n-1}$). Thus $\mathcal{E}^{n-1}$ is the only non-constant local system supported on $\mathcal{U}$. Consider the algebraic cycle $\Lambda_{\mathcal{Y}}$, and lift it to a class u supported on $\tilde{X} \times_{\mathbb{P}^1} \tilde{X}$; for $i > 0$ the morphisms

$$u^i L^i : \nu_* \nu^* \mathcal{R}^{n-1-i} \to \nu_* \nu^* \mathcal{R}^{n-1-i}$$

equal the identity on the geometric generic fibre, hence on each fibre, as the sheaves involved are constant. Specialising u on each geometric fibre $t \in \mathcal{U}_0$ we obtain the operators $\theta^i_{X_t}$ defined in Proposition 2.5(2), which proves $B(X_t)$ for each smooth fibre X_t of ρ. $\square$

We consider a suitable Lefschetz pencil for X, and prove the algebraicity of π_X^i for $i \leqslant n-2$.

It suffices to prove that the operators $\pi_{\tilde{X}}^i$ are algebraic for $i = 2, \ldots, n-2$, since $\pi^{2n-i} = {}^t\pi^i$ (Kleiman [16] showed already that π^0, π^1 are algebraic in general).

Lemma 7.3. *Assume $B(\mathcal{Y})$, ρ arbitrary. The projectors $\pi^{i-2,2}$ are algebraic for $i \leqslant n$, and so are $\pi^{i-2,0}$.*

Proof. We know that $k_* = \iota_* \oplus -h^*$. Let $i \leqslant n$. It is easy to see that

$$\pi^{i-2,2} = \left(\pi_X^i - p_X^i \oplus \pi_\Delta^{i-2}\right)\pi^{i-2,2}.$$

In fact, the following identity holds:

$$\pi^{i-2,2} = \left(\pi_X^i - p_X^i \oplus \pi_\Delta^{i-2}\right)\pi_\rho^{i-2}. \tag{19}$$

The above yields algebraicity of $\pi^{i-2,2}$ for $i \leqslant n$, and the algebraicity of the operator $\pi_\rho^{i-2} - \pi^{i-2,2} = \pi^{i-2,0}$ follows for $i \leqslant n$. The proof is now complete. □

Lemma 7.4. *Assumptions as in Proposition* 7.1. *The projectors $\pi_{\tilde{X}}^i$ are algebraic for $i \neq n-1$, $n, n+1$.*

Proof. This is immediate, since for $i \leqslant n-2$ the operator $\pi_{\tilde{X}}^i = \pi^{i,0} + \pi^{i-2,2}$ is algebraic by Lemma 7.3, as $\pi^{i,0} = \pi_\rho^i - \pi^{i,2}$ too is algebraic. □

Proof of Proposition 7.1. One need only remark that $p_X^i = \pi_X^i - (\pi_X^i - p_X^i)$ is algebraic for $i \leqslant n-2$, which holds by Lemma 7.2(iii). Proposition 7.1 is thus established. □

7.2. Proof of the Main Theorem

We finally prove the Main Theorem.

Assume that $B(\mathcal{Y})$ holds for $\mathcal{Y}$ the generic fibre of a Lefschetz fibration ρ of X satisfying condition **(A)**.

By Lemma 2.6, we have the following identity:

$$\iota^*\left(\Lambda_X - p_X^{n+1}\right) - \Lambda_Y \iota^* = \sum_{j=n+2}^{2n-2} \iota^* L^{j-n-1} p_X^j. \tag{20}$$

Assuming $B(Y)$, the l.h.s. of (20) is algebraic if and only if $\Lambda_X - p_X^{n+1}$ is. This follows from the identity

$${}^t\left[\iota^*\left(\Lambda_X - p_X^{n+1}\right)\right]\iota^*\left(\Lambda_X - p_X^{n+1}\right) = \Lambda_X - p_X^{n+1}. \tag{21}$$

The next step is to show that the r.h.s. of (20) is algebraic. This will follow from the next lemma ($j = 2n - i$).

Lemma 7.5. *Assume $B(\mathcal{Y})$ (and* **(A)***). For $0 \leqslant i \leqslant n-2$, the operator*

$$L^{n-i-1} p_X^{2n-i} = \Lambda_X{}^t p_X^i = f_* \Lambda_\rho f^{*t} p_X^i$$

is algebraic.

Proof. Let $i \leqslant n-2$; then p_X^i is algebraic by Proposition 7.1. Consider the subspace $W = (L^{n-1-i} P^i(\tilde{X}) \oplus L^{n-1-i} P^{i-2}(\Delta)) \cap F_\rho^2$, which agrees with the image of

$$k_* = \iota_* \oplus (-h^*) : L^{n-2-i} P^i(Y) \to L^{n-1-i} P^i(X) \oplus L^{n-2-i} P^i(\Delta)$$

and identifies with $H^2(L^{n-i-2}\mathcal{P}^i)$. The action of Λ_ρ on F^2 agrees with that of $\Lambda_{\mathcal{Y}}$ on $H^{*-2}(\mathcal{Y})_G$, which identifies with F^2 via the Gysin homomorphism, so

$$\Lambda_\rho\big(L^{n-i} P^i(X) \oplus 0\big) = W.$$

The first component of k_* is an isomorphism, and the second is injective, being bijective if $i < n-2$. On applying L, which coincides with L_ρ on F_ρ^2, we have an isomorphism

$$L : W \xrightarrow{\sim} L^{n-i} P^i(X) \oplus 0 \subset F_\rho^2;$$

the piece $L^{n-i} P^i(X) \oplus 0 = L_{\tilde{X}}^{n-i} P^i(\tilde{X})$ equals $k_* L_Y^{n-1-i} P^i(Y)$—see Corollary 4.7. L is thus an isomorphism between W and $L^{n-i} P^i(X) \oplus 0$. The identity

$$L_{\mathcal{Y}} \Lambda_{\mathcal{Y}} = 1_{\mathcal{Y}} - \sum_{i=0}^{n-1} p_{\mathcal{Y}}^i$$

[16, p. 372] translates by Proposition 6.2 into

$$L_\rho \Lambda_\rho = 1_{\tilde{X}} - \sum_{i=0}^{n-1} p_\rho^i,$$

thus showing that Λ_ρ defines the inverse isomorphism to $L : W \to L^{n-i} P^i(X) \oplus 0$ (any lifting of $\Lambda_{\mathcal{Y}}$ would do, since we apply these operators on F^2); transposing yields $L\Lambda_\rho = 1$ when restricted to $L^{n-i} P^i(X) \oplus 0$. Taking the X-component yields the inverse

$$L^{n-i} P^i(X) \oplus 0 \to W \to L^{n-1-i} P^i(X)$$

of L, which coincides with $\Lambda_X | L^{n-i} P^i(X)$—here we have used that L agrees with L_ρ on F_ρ^2, and that for $i \leqslant n-1$, p_ρ^i acts as 0 on $H^j(\tilde{X})$ for $j \geqslant n+2$. We have thus proven that $\Lambda_X^t p_X^{n-i} = L^{n-1-i} p^{n+i} = f_* \Lambda_\rho f^{*t} p_X^{n-i}$ is algebraic by Propositions 6.2 and 7.1. □

Lemma 7.6. *Assuming the hypotheses of the Main Theorem, the operator*

$$\Lambda_X \pi_X^i = \pi_X^{i-2} \Lambda_X = \Lambda_X\big(\pi_X^i - p_X^i\big)$$

is algebraic for $i \leqslant n$. The operator $(\pi_X^{n-1} - p_X^{n-1})\Lambda_X = (\Lambda_X - p_X^{n+1})\pi_X^{n+1}$ is algebraic.

Proof. The identities are clear; let us prove algebraicity of the above operators. By Lemmas 2.6 and 7.2(iii) we have

$$\begin{aligned}\Lambda_X\big(\pi_X^i - p_X^i\big) &= \Lambda_X \iota_* \Lambda_Y \pi_Y^i \iota^* = \iota_* \Lambda_Y^2 \pi_Y^i \iota^* + \left(p_X^{n+1} + \sum_{j=n+2}^{2n-2} p_X^j L^{j-n+1}\right) \iota_* \Lambda_Y \pi_Y^i \iota^* \\ &= \iota_* \Lambda_Y^2 \pi_Y^i \iota^* + \left(\sum_{j=n+2}^{2n-2} p_X^j L^{j-n+1}\right) \iota_* \Lambda_Y \pi_Y^i \iota^*,\end{aligned}$$

which is algebraic for $i \leqslant n+1$ by Lemma 7.5, thereby establishing the lemma. □

Proof of Main Theorem. Under the hypotheses of this section, the operator $\Lambda_X - p_X^{n+1}$ is algebraic.

Indeed, we have proven in Lemma 7.6 that $\pi_X^{n-2}\Lambda_X = \Lambda_X \pi_X^n$ and $\pi_X^{n-3}\Lambda_X = \Lambda_X \pi_X^{n-1}$ are algebraic, as well as the algebraicity of $(\Lambda_X - p_X^{n+1})\pi_X^{n+1}$. It now remains to establish the algebraicity of $\Lambda_X \sum_{i=n+2}^{2n} \pi_X^i$. Again, Lemma 7.6 shows that, for $r \geqslant 2$, the operator ${}^t(\Lambda_X \pi_X^{n+r}) = \pi_X^{n-r}\Lambda_X$ is algebraic. On the other hand, $(\Lambda_X - p_X^{n+1})\pi_X^{n+1} = \Lambda(\pi_X^{n+1} -{}^t p_X^{n-1})$. It follows that the operator

$$\Lambda_X - p_X^{n+1} = \Lambda_X\left(\sum_{k=0}^{n} \pi_X^i + \big(\pi_X^{n+1} -{}^t p_X^{n-1}\big) + \sum_{k=n+2}^{2n} \pi_X^i\right)$$

is algebraic, thus establishing the Main Theorem. □

Remark 7.7. Alternatively, the Main Theorem follows from Lemma 7.5 and identities (20), (21).

7.3. Comments on the Main Theorem

This is a mere rephrasing of the Main Theorem. The details are left to the reader.

Restatement of the Main Theorem. In the language of Proposition 2.5, our Main Theorem states precisely that θ^i is induced by an algebraic cycle (to wit $(\Lambda - p^{n+1})^{n-i}$) for $i \leqslant n-2$. By the proof of [16, Lemma 2.4], one may derive expressions of π_X^i for $i \neq n-1, n, n+1$ showing algebraicity of these projectors.

7.4. The operator p_X^{n+1}

This section is a complement to the Main Theorem. The operator p_X^{n+1} (and so $\iota^* p_X^{n+1}$) is of central importance in the Lefschetz theory of X.

Lemma 7.8. *Assume $B(\mathcal{Y})$ for $\mathcal{Y}$ the general fibre of a Lefschetz pencil of X satisfying* **(A)**. *The algebraicity of p_X^{n-1} implies that of p_X^n and the conjecture $C(X)$.*

Proof. We know that

$$^t p^{n-1} + p^n + p^{n-1} = 1 - \left(\sum_{|i-n \geqslant 2|} \pi^i\right) - \left(\pi^n - p^n\right) - \left(\pi^{n-1} - p^{n-1}\right) -^t \left(\pi^{n-1} - p^{n-1}\right).$$

The lemma follows from Lemma 7.2 and Proposition 7.1. □

Lemma 7.9. *Let X be a projective smooth, n-dimensional variety. The operator $\iota^* p^{n+1}$ is algebraic if p^{n+1} is.*

Proof. The result stems from the identity

$$^t\left(\iota^* p^{n+1}\right)\iota^* p^{n+1} = p^{n+1} L p^{n+1} = p^{n+1}. \qquad \square \tag{22}$$

Enclosed in p^{n+1} is information about the space of vanishing cycles, and also p^{n-1}. In particular, the algebraicity of p^{n+1} allows us to speak of the **motive of vanishing cycles**.

Proposition 7.10. *With the above notations, the following statements hold.*

1. *The algebraicity of p_X^{n+1} implies that of p_X^{n-1}.*
2. *Let $V(Y)$ be the space of vanishing cycles of a smooth hyperplane section Y, and $e_{V(Y)}$ be the orthogonal projection $H^*(Y) \twoheadrightarrow V(Y) \hookrightarrow H^*(Y)$. Then*

$$p_Y^{n-1} - e_{V(Y)} = \iota^* p^{n+1} \iota_*.$$

3. *If $B(Y)$ holds and p^{n+1} is algebraic, then so is $e_{V(Y)}$.*

The following result illustrates the difficulty in constructing Λ_X and p^{n+1}.

Proposition 7.11. *Let ρ be an arbitrary Lefschetz fibration. There is no correspondence u supported on $(\rho \times \rho)^{-1}(E)$ for E a 1-dimensional subscheme of $\mathbb{P}^1 \times \mathbb{P}^1$ such that $f_* u f^*$ induces p^{n+1} on $LP^{n-1}(X)$.*

Proof. We know that $LP^{n-1}(X) \oplus 0 \subset F^2$. Proposition 4.16 shows that any correspondence u of degree -1 supported on $(\rho \times \rho)^{-1}(E)$ for E a proper subscheme satisfies $uLP^{n-1}(X) \oplus 0 \subset F^2 H^{n-1}(\tilde{X}) \subset LH^{n-3}(X) \oplus H^{n-3}(\Delta)$. We conclude that $f_* u f^* LP^{n-1}(X) \cap P^{n-1}(X) = 0$, which proves the proposition. □

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